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A Note on Trace-Differentiation and the Ω-operator

By A. C. Aitken

(*Received 27th January* 1949. *Read 4th March* 1949.)

1. In theory of polarizing operators in invariants the operator

$$\Omega = \left[\frac{\partial}{\partial x_{ij}} \right], \tag{1}$$

where $X = [x_{ij}]$ is an $n \times n$ matrix of n^2 independent elements x_{ij}, holds an important place. Acting upon particular scalar functions of X, namely the spur or trace of powers of X, or of polynomials or rational functions of X with scalar coefficients, it exhibits (Turnbull, 1927, 1929, 1931) an exact analogy with results in the ordinary differentiation of the corresponding functions of one scalar variable. Turnbull denotes this operation of trace-differentiation under Ω by Ω_x; and we shall follow him. Our purpose is to show how, with a suitably modified Ω, the results may be extended to the case of symmetric matrices $X = X'$ having $\frac{1}{2}n(n+1)$ independent elements.

Such extensions forced themselves on the notice of the author in some work (Aitken, 1946) on the estimation of statistical parameters. At the same time and independently Garding was using the determinant of the same modified Ω to obtain the analogue (Garding, 1947) of Cayley's determinantal theorem (cf. Turnbull, 1928, p. 114)

$$|\ \Omega\ |\ X\ |^r = r(r+1)\,(r+2)\,\ldots\,(r+n-1)\ |\ X\ |^{r-1}, \tag{2}$$

namely

$$|\ \Omega\ |\ X\ |^r = r(r+\tfrac{1}{2})\,(r+1)\,\ldots\,(r+\tfrac{1}{2}n-\tfrac{1}{2})\ |\ X\ |^{r-1}, \tag{3}$$

where in (3) the matrix X is symmetric and Ω is modified.

2. We first establish the result 1 (2), following one of Turnbull's proofs, when r is a positive integer. The extension to the case of negative powers of X and to rational functions will then be taken in hand. The corresponding results for a symmetric matrix X follow without difficulty.

It is of advantage to use tensor notation and the summation convention. We denote x_{ij} by x^i_j, and we agree that the summation convention shall hold for repeated Greek indices, but not for italic.

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where in (3) the matrix X is symmetric and Ω is modified.

2. We first establish the result 1 (2), following one of Turnbull's proofs, when r is a positive integer. The extension to the case of negative powers of X and to rational functions will then be taken in hand. The corresponding results for a symmetric matrix X follow without difficulty.

It is of advantage to use tensor notation and the summation convention. We denote x_{ij} by x^i_j, and we agree that the summation convention shall hold for repeated Greek indices, but not for italic.

Thus the $(i, j)^{\text{th}}$ element in X^r, the i^{th} diagonal element, and the trace tr X^r, are respectively

$$x_\alpha^i x_\beta^\alpha x_\gamma^\beta \dots x_j^\pi, \qquad x_\alpha^i x_\beta^\alpha x_\gamma^\beta \dots x_i^\pi, \qquad \text{tr } X^r = x_\rho^\rho x_\alpha^\alpha x_\beta^\beta x_\gamma^\gamma \dots x_\rho^\pi, \tag{1}$$

the last being an aggregate of n^r terms, the row and column indices of the factors in each term being linked, as shown, in a closed cycle of r indices. The sub-aggregate of terms in tr X^r having x_i^j for first factor is $x_i^j x_\beta^i x_\gamma^\beta \dots x_j^\pi$.

To differentiate partially with respect to x_i^j is to delete the factor x_i^j once wherever it occurs, and to sum all terms so obtained, each such term being a product of $r - 1$ elements. Performing this deletion of x_i^j in each of the r factors we obtain r times in all, in virtue of the cyclic symmetry, the expression $x_\beta^i x_\gamma^\beta \dots x_j^\pi$, that is, the $(i, j)^{\text{th}}$ element in X^{r-1}. We thus have, for this first case of a general matrix X,

$$\Omega_s X^r = r X^{r-1}, \tag{2}$$

and the extension

$$\Omega_s f(X) = f'(X), \tag{3}$$

where $f(X)$ is a polynomial in X with scalar coefficients, is immediate.

Further extensions, such as

$$\Omega_s A X^r = A X^{r-1} + X A X^{r-2} + X^2 A X^{r-3} + \dots + X^{r-1} A, \tag{4}$$

admit of similar proof. We have only to notice that the trace of AX^r is $a_\alpha^\sigma x_\beta^\alpha x_\gamma^\beta \dots x_\sigma^\rho$, to carry out deletion of x_i^j in each of the r positions and to observe the fact of cyclic symmetry. More generally, and for similar reasons, the result of Ω_s acting on $AX^p BX^q CX^r \dots X^t K$ is given by the rule: Arrange the factors of the operand, X^p being written out as $XXX \dots X$ and so on, in all the possible cyclic orders in which X appears as first factor. In each of these, delete this first X. The sum of the resulting products of matrices is the desired trace-derivative.

3. We now extend the result 2 (2) to the case X^{-r}, where r is a positive integer. The $(i, j)^{\text{th}}$ element in X^{-r} is then $X_\alpha^j X_\beta^\alpha X_\gamma^\beta \dots X_i^\pi$ and so

$$\text{tr } X^{-r} = X_\alpha^\rho X_\beta^\alpha X_\gamma^\beta \dots X_\rho^\pi. \tag{1}$$

The effect of $\dfrac{\partial}{\partial x_{ji}}$ is, as usual, to operate on each factor as if the rest were constant, and to add the results. Denoting the cofactor of x_i^j in $| X_\beta^\alpha |$ by $| X_{\beta i}^{\alpha j} |$, and observing that $X_\beta^\alpha = | X_\beta^\alpha | / | X |$,

we have

$$\frac{\partial}{\partial x_{ji}} X^{\alpha}_{\beta} = \begin{vmatrix} X^{\alpha j}_{\beta i} & X^{\alpha}_{\beta} \\ X^{j}_{i} & X \end{vmatrix} \div |X|^2. \tag{2}$$

Now the numerator is an "extensional" determinant, and so (2) is readily evaluated as

$$-X^{\alpha}_{i} \dots X^{j}_{\beta}. \tag{3}$$

Thus whereas in case of X^r trace-differentiation has the effect of *deleting* in r ways a link from a chain, in the case of X^{-r} it has the effect of *inserting* a link, again in r ways, with change of sign. For example one of the r terms in

$$\frac{\partial}{\partial x_{ji}} X^{\rho}_{\alpha} X^{\alpha}_{\beta} X^{\beta}_{\gamma} \dots X^{\pi}_{\rho} \qquad \text{is} \qquad -X^{\rho}_{a} X^{a}_{i} X^{j}_{\beta} X^{\beta}_{\gamma} \dots X^{\pi}_{\rho},$$

namely a term in the $(i, j)^{\text{th}}$ element of $-X^{-r-1}$. Thus, again in virtue of cyclic symmetry, we have

$$\Omega_s X^{-r} = -rX^{-r-1}. \tag{4}$$

It is easy to deduce a result analogous to 2 (4) for the effect of Ω_s on AX^{-r}.

4. Next let X be a *symmetric* matrix $X = X'$, the elements being $\frac{1}{2}n(n+1)$ independent variables. The main lines of the reasoning are the same; we have only to note the alteration caused by the fact that $x_{ji} = x_{ij}$. First, deletions and insertions with respect to x^{i}_{i} will be as before. With respect to x^{j}_{i}, $i \neq j$, to every term obtained by a deletion (or insertion) with respect to it there will correspond, by symmetry, a duplicate term obtained by a deletion (or insertion) with respect to x^{i}_{j}. Thus the sets of terms arising from the non-diagonal operators in Ω_s will appear twice. We can compensate for this by using a suitably modified Ω, namely

$$\Omega = \left[\epsilon_{ji} \frac{\partial}{\partial x_{ji}}\right], \qquad \epsilon_{ii} = 1, \ \epsilon_{ij} = \epsilon_{ji} = \tfrac{1}{2}, \ i \neq j, \tag{1}$$

and now we have, for a symmetric matrix X,

$$\Omega_s X^r = rX^{r-1}, \qquad \Omega_s X^{-r} = -rX^{-r-1}. \tag{2}$$

We may add, for completeness, the result

$$\Omega_s \log X = \Omega |X| = X^{-1}, \qquad |X| \neq 0, \tag{3}$$

easily proved in an elementary way.

The case of a rational function $f(X)\{g(X)\}^{-1}$ can be treated by resolution into partial fractions and use of the result, an immediate consequence of 2 (2),

$$\Omega_s (X + aI)^{-r} = -r(X + aI)^{-r-1}, \tag{4}$$

where a is a scalar constant.

Thus, both in the general and in the symmetric case, we have theorems, in trace-differentiation of polynomials and rational functions of X, resembling in all respects the familiar theorems of ordinary differentiation.

5. It is of interest, even though the results are rather formal, to note that with a further slight modification of Ω we may apply trace-differentiation to skew symmetric matrices. We then have

$$X' = -X, \text{ that is, } x_{ii} = 0,\ x_{ji} = -x_{ij},$$

and it is readily shown that if

$$\Omega = \left[\epsilon_{ji} \frac{\partial}{\partial x_{ji}}\right],\ \epsilon_{ii} = 0,\ \epsilon_{ij} = \epsilon_{ji} = \tfrac{1}{2},\ i \neq j,$$

then trace-differentiation preserves those formal analogies which it has been the object of this note to communicate. It must be observed, however, that odd powers of X are debarred, since the trace is then identically zero. The rule applies to rational functions in which only even powers appear.

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[1] In (5) and (6) on p. 42 of this paper, $-rX^{-r+1}$ should be replaced by $-rX^{-(r+1)}$.

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Matrix Differentiation of S-Functions

By H. O. Foulkes

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1. It has been shown (1; 2, 136) that if S_r, a_r, h_r denote respectively the symmetric functions $\Sigma\lambda_i^r$, $\Sigma\lambda_1\lambda_2\ldots\lambda_r$, and the homogeneous product sum of degree r of the latent roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $X = [x_{ij}]$, then

$$\Omega S_r = rX^{r-1}, \tag{1}$$

$$\Omega a_r = (-1)^{r-1}\{X^{r-1} - a_1 X^{r-2} + a_2 X^{r-3} - \ldots + (-1)^{r-1} a_{r-1}\}, \tag{2}$$

$$\Omega h_r = X^{r-1} + h_1 X^{r-2} + h_2 X^{r-3} + \ldots + h_{r-1}, \tag{3}$$

where $\Omega = \left[\dfrac{\partial}{\partial x_{ji}}\right]$.

The symmetric functions known as Schur-functions, or S-functions, are defined (3, 82) in terms of group characters and are each associated with a partition of a number. In particular the S-functions denoted by $\{1^r\}$ and $\{r\}$ are respectively a_r and h_r. Hence (2) and (3) are particular cases of a more general expression for $\Omega\{\lambda\}$ where $\{\lambda\}$ is the S-function corresponding to any partition (λ). In this note an expression for $\Omega\{\lambda\}$ is obtained in the form of a polynomial in X with coefficients which are linear functions of S-functions, and which can be found without recourse to tables of group characters.

In addition, generalisations of Turnbull's (4) theorems

$$\Omega^2 h_{r+1} = (n+r)\,\Omega h_r,$$

and

$$\Omega^2 a_{r+1} = -(n-r)\,\Omega a_r,$$

are also obtained. It is found that in the general theorems h_{r+1} and a_{r+1} have to be replaced by certain linear functions of S-functions, and the iterative property begins with a higher power of Ω.

2. Since
$$\frac{\partial S_r^a}{\partial x_{ji}} = \frac{\partial S_r^a}{\partial S_r}\frac{\partial S_r}{\partial x_{ji}}$$

then
$$\Omega S_r^a = \frac{\partial S_r^a}{\partial S_r}\,\Omega S_r.$$

Similarly $\Omega(S_r^a S_s^b S_t^c \ldots)$

$$= \frac{\partial(S_r^a S_s^b S_t^c \ldots)}{\partial S_r}\Omega S_r + \frac{\partial(S_r^a S_s^b S_t^c \ldots)}{\partial S_s}\Omega S_s + \frac{\partial(S_r^a S_s^b S_t^c \ldots)}{\partial S_t}\Omega S_t + \ldots.$$

By definition,

$$\{\lambda\} = \frac{1}{m!}\sum_\rho h_\rho \chi_\rho^{(\lambda)} S_1^a S_2^b \ldots$$

where h_ρ is the order of the class ρ determined by the partition $1^a 2^b \ldots$ of m, and $\chi_\rho^{(\lambda)}$ is the characteristic of the class ρ in the irreducible representation of the symmetric group associated with the partition (λ) of m. Hence

$$\Omega\{\lambda\} = \frac{\partial\{\lambda\}}{\partial S_1}\Omega S_1 + \frac{\partial\{\lambda\}}{\partial S_2}\Omega S_2 + \ldots + \frac{\partial\{\lambda\}}{\partial S_n}\Omega S_n$$

$$= \frac{\partial\{\lambda\}}{\partial S_1} + 2\frac{\partial\{\lambda\}}{\partial S_2}X + 3\frac{\partial\{\lambda\}}{\partial S_3}X^2 + \ldots + n\frac{\partial\{\lambda\}}{\partial S_n}X^{n-1}.$$

The coefficients of the various powers of X could be found by differentiation and expressed in terms of S-functions if the appropriate tables of group characters are available. They can, however, be found more easily without the use of the tables. Any operator $\partial/\partial S_r$ can be written as

$$\frac{1}{j}\left[\chi_r^{(\lambda_1)} D_{\lambda_1} + \chi_r^{(\lambda_2)} D_{\lambda_2} + \ldots \chi_r^{(\lambda_j)} D_{\lambda_j}\right]$$

where $\chi_r^{(\lambda)}{}_i$ is the characteristic of the class (r) of the symmetric group of order $r!$ corresponding to the partition (λ_i), and D_{λ_i} is an operator defined as (5)

$$\sum_\rho \chi_\rho^{(\lambda_i)} \frac{1}{a!\,b!\ldots}\frac{\partial^{a+b+\ldots}}{\partial S_1^a \partial S_2^b \ldots}.$$

It so happens that the characteristics of the class (r) are $(-1)^{k-1}$ for every partition $(r-k+1, 1^{k-1})$, $k = 1, 2, \ldots, r$, and are zero for all other partitions (6, 134). Hence

$$r\frac{\partial}{\partial S_r} = D_r - D_{r-1,1} + D_{r-2,1^2} - D_{r-3,1^3} + \ldots + (-1)^{r-1} D_{1^r}.$$

The effect of D_μ on $\{\lambda\}$ is (5)

$$D_\mu\{\lambda\} = \sum g_{\mu\nu\lambda}\{\nu\}$$

where $g_{\mu\nu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\mu\}\{\nu\}$. Hence the coefficients in

$\Omega\{\lambda\}$ can be found from a knowledge of products of S-functions, without using the tables of group characters. For example,

$$\Omega\{2^2 1^2\} = D_1\{2^2 1^2\} + (D_2 - D_{1^2})\{2^2 1^2\} X + (D_3 - D_{21} + D_{1^3})\{2^2 1^2\} X^2$$
$$+ (D_4 - D_{31} + D_{21^2} - D_{1^4})\{2^2 1^2\} X^3 + (D_5 - D_{41} + D_{31^2} - D_{21^3} + D_{1^5})\{2^2 1^2\} X^4$$
$$= (\{21^3\} + \{2^2 1\}) - (\{1^4\} + \{2^2\}) X + \{2\} X^3 - \{1\} X^4.$$

When the number of parts in a partition exceeds n, the corresponding S-function vanishes identically (3, 91), and the expression for $\Omega\{\lambda\}$ then reduces to a polynomial identity in X, which must of course contain the characteristic function of X as a factor. Thus in the above example, if $n = 3$, then

$$\{1\} X^4 - \{2\} X^3 + \{2^2\} X - \{2^2 1\} = 0,$$

which is equivalent to

$$(X^3 - \{1\} X^2 + \{1^2\} X - \{1^3\})(\{1\} X + \{1^2\}) = 0.$$

3. To extend Turnbull's results

$$\Omega^2\{p\} = (n + p - 1)\,\Omega\{p - 1\}, \tag{4}$$

$$\Omega^2\{1^q\} = -(n - q + 1)\,\Omega\{1^{q-1}\} \tag{5}$$

a stage further, operate twice with Ω on the relation

$$\{p 1^q\} + \{p + 1, 1^{q-1}\} = \{p\}\{1^q\},$$

obtaining

$$\Omega\{p 1^q\} + \Omega\{p + 1, 1^{q-1}\} = \{p\}\,\Omega\{1^q\} + \{1^q\}\,\Omega\{p\}$$

and

$$\Omega^2\{p 1^q\} + \Omega^2\{p + 1, 1^{q-1}\} = 2\Omega\{p\}\,\Omega\{1^q\} + \{p\}\,\Omega^2\{1^q\} + \{1^q\}\,\Omega^2\{p\}$$
$$= 2\Omega\{p\}\,\Omega\{1^q\} - (n - q + 1)\{p\}\,\Omega\{1^{q-1}\} + (n + p - 1)\{1^q\}\,\Omega\{p - 1\}. \tag{6}$$

When $q = 1$, this becomes

$$\Omega^2\{p1\} = 2\Omega\{p\} + (n + p - 1)\{1\}\,\Omega\{p - 1\} - (n + p)\,\Omega\{p\}$$
$$= (n + p - 1)\{1\}\,\Omega\{p - 1\} - (n + p - 2)\,\Omega\{p\}.$$

Again applying Ω, we have

$$\Omega^3\{p1\} = (n + p - 1)[\Omega\{p - 1\} + (n + p - 2)\{1\}\,\Omega\{p - 2\}]$$
$$- (n + p - 2)(n + p - 1)\,\Omega\{p - 1\}$$
$$= (n + p - 1)[(n + p - 2)\{1\}\,\Omega\{p - 2\} - (n + p - 3)\,\Omega\{p - 1\}]$$
$$= (n + p - 1)\,\Omega^2\{p - 1, 1\}.$$

Hence (4), which may be written as

$$\Omega^r\{p\} = (n+p-1)\,\Omega^{r-1}\{p-1\}, \qquad p>0,\ r\geqq 2,$$

has as an extension the theorem that

$$\Omega^r\{p1\} = (n+p-1)\,\Omega^{r-1}\{p-1,\,1\}, \qquad p>1,\ r\geqq 3. \tag{7}$$

Similarly, by taking $p=1$ in (6), we have

$$\Omega^2\{2,\,1^{q-1}\} = (n-q+2)\,\Omega\{1^q\} - (n-q+1)\{1\}\,\Omega\{1^{q-1}\}$$

and

$$\Omega^3\{2,\,1^{q-1}\} = -(n-q+2)(n-q+1)\,\Omega\{1^{q-1}\}$$
$$-(n-q+1)\,[\Omega\{1^{q-1}\} - (n-q+2)\{1\}\,\Omega\{1^{q-2}\}]$$
$$= -(n-q+1)\,[(n-q+3)\,\Omega\{1^{q-1}\} - (n-q+2)\{1\}\,\Omega\{1^{q-2}\}]$$
$$= -(n-q+1)\,\Omega^2\{2,\,1^{q-2}\}.$$

If $\{2,\,1^{q-1}\}$ is written as $\{1^q\,1\}$ in order to bring the form of the result into correspondence with (7), then (5), written as

$$\Omega^r\{1^q\} = -(n-q+1)\,\Omega^{r-1}\{1^{q-1}\}, \qquad q>0,\ r\geqq 2,$$

has as an extension the theorem that

$$\Omega^r\{1^q\,1\} = -(n-q+1)\,\Omega^{r-1}\{1^{q-1}1\}, \qquad q>1,\ r\geqq 3. \tag{8}$$

Taking $q=2$ and $p=2$ in (6) does not lead to corresponding expressions for $\Omega^r\{p1^2\}$ and $\Omega^r\{1^q1^2\}$, but the simple recursive property shown in (4), (5), (7), (8) appears again in the results of the next section, in which a new type of operand is considered.

4. Consider the expressions defined as

$$(p,\,0) = \{p\}, \qquad p>0,$$
$$(p,\,1) = \{p\,1\}, \qquad p>1,$$
$$(p,\,2) = \{p\,2\} + \{p\,1^2\}, \qquad p>2,$$
$$(p,\,3) = \{p\,3\} + 2\{p\,21\} + \{p1^3\}, \qquad p>3,$$
$$(p,\,4) = \{p\,4\} + 3\{p\,31\} + 2\{p\,2^2\} + 3\{p\,21^2\} + \{p\,1^4\}, \qquad p>4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

in which the coefficient of $\{p\,\lambda_1\,\lambda_2\,\lambda_3\ldots\}$ in $(p,\,m)$ is the characteristic of the class (1^m) in the irreducible representation of the symmetric group of order $m!$ corresponding to the partition $(\lambda_1\,\lambda_2\,\lambda_3\ldots)$ of m. The set of coefficients can be written down at once from the character table, as they constitute the first column of the table as usually presented (3).

These expressions are such that

$$\{1\}(p, m) = (p, m+1) + (p+1, m).$$

Hence

$$\Omega(p, m+1) = (p, m) + \{1\}\Omega(p, m) - \Omega(p+1, m),$$

$$\Omega^2(p, m+1) = 2\Omega(p, m) + \{1\}\Omega^2(p, m) - \Omega^2(p+1, m),$$

$$\Omega^3(p, m+1) = 3\Omega^2(p, m) + \{1\}\Omega^3(p, m) - \Omega^3(p+1, m),$$

.....

$$\Omega^r(p, m+1) = r\Omega^{r-1}(p, m) + \{1\}\Omega^r(p, m) - \Omega^r(p+1, m).$$

From (4), $\Omega^2(p, 0) = (n+p-1)\Omega(p-1, 0)$, and so for $m = 0$

$$\Omega^2(p, 1) = (n+p-1)\{1\}\Omega(p-1, 0) - (n+p-2)\Omega(p, 0),$$

$$\begin{aligned}\Omega^3(p, 1) &= (n+p-1)[(n+p-2)\{1\}\Omega(p-2, 0) - (n+p-3)\Omega(p-1, 0)] \\ &= (n+p-1)\Omega^2(p-1, 1),\end{aligned}$$

which is (7). For $m = 1$,

$$\begin{aligned}\Omega^4(p, 2) &= 4\Omega^3(p, 1) + \{1\}\Omega^4(p, 1) - \Omega^4(p+1, 1) \\ &= 3\Omega^3(p, 1) + (n+p-1)\{1\}\Omega^3(p-1, 1) - (n+p)\Omega^3(p, 1) + \Omega^3(p, 1) \\ &= (n+p-1)[3\Omega^2(p-1, 1) + \{1\}\Omega^3(p-1, 1) - \Omega^3(p, 1)] \\ &= (n+p-1)\Omega^3(p-1, 2).\end{aligned}$$

The generalisation is now evident and the proof follows by induction. If $\Omega^r(p, m) = (n+p-1)\Omega^{r-1}(p-1, m)$ for $r \geqq m+2$, then

$$\begin{aligned}\Omega^r(p, m+1) &= (r-1)\Omega^{r-1}(p, m) + \{1\}\Omega^r(p, m) - \Omega^r(p+1, m) \\ &\qquad + \Omega^{r-1}(p, m) \\ &= (n+p-1)[(r-1)\Omega^{r-2}(p-1, m) + \{1\}\Omega^{r-1}(p-1, m) - \Omega^{r-1}(p, m)] \\ &\qquad \text{for } r-1 \geqq m+2, \\ &= (n+p-1)\Omega^{r-1}(p-1, m+1), \qquad \text{for } r \geqq m+3.\end{aligned}$$

5. A corresponding result can be obtained for the conjugate partitions. For convenience in notation write the conjugate of $\{q\,\lambda_1\,\lambda_2\ldots\}$ as $\{1^q\mu_1\,\mu_2\ldots\}$, where $\{\mu_1\,\mu_2\ldots\}$ and $\{\lambda_1\,\lambda_2\ldots\}$ are conjugates. Then the expressions

$$(1^q, 0) = \{1^q\}, \qquad q > 0,$$

$$(1^q, 1) = \{1^q\,1\}, \qquad q > 1,$$

$$(1^q, 2) = \{1^q\,2\} + \{1^q\,1^2\}, \qquad q > 2,$$

$$(1^q, 3) = \{1^q\,3\} + 2\{1^q\,21\}, + \{1^q\,1^3\}, \qquad q > 3,$$

$$(1^q, 4) = \{1^q\,4\} + 3\{1^q\,31\} + 2\{1^q\,2^2\} + 3\{1^q\,21^2\} + \{1^q\,1^4\}, \qquad q > 4,$$

.....

are such that

$$\{1\}(1^q, m) = (1^q, m+1) + (1^{q+1}, m).$$

Hence

$$\Omega^r(1^q, m+1) = r\,\Omega^{r-1}(1^q, m) + \{1\}\Omega^r(1^q, m) - \Omega^r(1^{q+1}, m).$$

Assume

$$\Omega^r(1^q, m) = -(n-q+1)\,\Omega^{r-1}(1^{q-1}, m) \qquad \text{for } r \geqq m+2.$$

Then

$$\begin{aligned}\Omega^r(1^q, m+1) &= (r-1)\,\Omega^{r-1}(1^q, m) + \{1\}\Omega^r(1^q, m) \\ &\qquad - \Omega^r(1^{q+1}, m) + \Omega^{r-1}(1^q, m) \\ &= -(n-q+1)\,[(r-1)\,\Omega^{r-2}(1^{q-1}, m) + \{1\}\Omega^{r-1}(1^{q-1}, m) \\ &\qquad - \Omega^{r-1}(1^q, m)] \qquad \text{for } r-1 \geqq m+2 \\ &= -(n-q+1)\,\Omega^{r-1}(1^{q-1}, m+1), \qquad \text{for } r \geqq m+3.\end{aligned}$$

Since $m = 0$ gives

$$\Omega^r(1^q, 0) = -(n-q+1)\,\Omega^{r-1}(1^{q-1}, 0) \qquad \text{for } r \geqq 2,$$

which is known to be true by (5), then the result is proved by induction.

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Note on γ-Matrices Efficient at an Isolated Point

By P. Vermes

(*Received 2nd September* 1948. *Read 5th November* 1948.)

In a recent paper [1] γ-matrices were constructed summing the binomial series at isolated points *to its right value.* It is assumed that the reader is able to refer to this paper and is familiar with its notation and terminology.

The object of this note is to give the construction of γ-matrices which sum the binomial series at an isolated point *to an arbitrary given value,* while inside the circle of convergence the generalised sum is necessarily the right value.

We first consider the series Σz^k at $z = z_0$, where $| z_0 | > 1$. If a is an arbitrary complex number, and $b = a - 1/(1 - z_0)$, we construct the matrix $G \equiv (g_{n,k})$, $n, k = 0, 1, 2, \ldots,$

given by $g_{n,k} = 1$ for $k < n$, $= 1/(1 - z_0)$ for $k = n$, $= b/2^{k-n} z_0^{k}$ for $k > n$.

Obviously G is a γ-matrix, since $g_{n,k} \to 1$ as $n \to \infty$, and

$$\sum_k | g_{n,k} - g_{n,k+1} | < 1 + 2/|1 - z_0| + 2|b| \Sigma |2z_0|^{-k} \quad \text{for every } n.$$

Applying G to the series Σz_0^k, we have

$$S_n = \sum_k g_{n,k} z_0^k = 1 + z_0 + z_0^2 + \ldots + z_0^{n-1} + z_0^n/(1 - z_0) + b/2 + b/4 + \ldots$$

$$= 1/(1 - z_0) + b = a,$$

so that $S_n \to a$.

For other values of z, $S_n(z) = \sum_k g_{n,k} z^k$ converges only if $|z| < 2|z_0|$, and then

$$S_n(z) = \frac{1}{1 - z} + \frac{(z_0 - z) z^n}{(1 - z)(1 - z_0)} + \frac{b z^{n+1} z_0}{(2z_0 - z) z_0^{n+1}},$$

which tends to a limit only if $z = z_0$ or $| z | < 1$.

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[1] P. Vermes, "On γ-matrices and their application to the binomial series," *Proc. Edinburgh Math. Soc.* (2), **8** (1947), 1-13 (11-13).

In the above construction we used the matrix $H(1, z_0)$ of the previous paper, replacing its zero elements by suitable numbers. Given the series $\Sigma c_k z^k$ representing $(1-z)^{-p}$, where p is a positive integer, we replace the zero elements of the matrix $H(p, z_0)$ in a similar way. We then have

$$g_{n,k} = (1-z_0)^{-p} \sum_{j=0}^{n-k} \binom{p}{j} (-z_0)^j \text{ for } k \leqq n, \quad = b/c_k z_0^{k-n} \text{ for } k > n,$$

where $b = a - 1/(1-z_0)^p$, and G has the required properties.

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Theorems Relating to Quadratic Forms and their Discriminant Matrices

By S. VAJDA

(*Received 16th May 1949. Read 4th June 1949.*)

In a paper read before the Research Branch of the Royal Statistical Society (Ref. 1, p. 150) the following case was considered: Let the expression $\sum_{i=1}^{n} (y_i - ca_i)^2$ be given; introduce, for c, a linear form in y_j, viz. $c = \sum_{j=1}^{n} b_j y_j$ and obtain

$$\sum_{i=1}^{n} (y_i - a_i \sum_{j=1}^{n} b_j y_j)^2. \tag{1}$$

If the y_i are sample values from a normal population with unit variance, then it is known (Ref. 2) that (1) is distributed as $\sum_i l_i z_i$, where z_i varies as chi-squared with one degree of freedom and the l_i are the latent roots of the matrix of the quadratic form. If these latent roots are f times unity and $n-f$ times zero, then this reduces to a chi-squared distribution with f degrees of freedom. It was shown that $\sum_i a_i b_i = 1$ was a necessary but not a sufficient condition for such a distribution with $f = n - 1$ to arise and it was mentioned that the latter is the case if we take the least-square solution for the b_i. As will be seen from what follows, the least-square solution is, in fact, the only choice for the b_i which leads to a chi-squared distribution with $n - 1$ degrees of freedom.

In the following lines we investigate problems related to a more general expression, viz., in matrix notation,

$$(y - Ac)'(y - Ac) \tag{2}$$

where y is a column matrix of order n, c a column matrix of order m and A is a matrix of order $n \times m$ and of rank m. We introduce $c = By$, where B is a matrix of order $m \times n$, thus obtaining

$$(y - ABy)'(y - ABy) = y'M'My. \tag{3}$$

The matrix of the latter quadratic form is seen to be $M'M$, where M is defined as $I_n - AB$.

The rank of M is the same as that of $M'M$, and the first theorem which we shall prove states that

The rank of M (and hence that of $M'M$) cannot be less than $n-m$.

We show this by applying a transformation $P^{-1}MP$ with non-singular P, so that the rank of the transformed matrix is the same as that of M.

We write A as $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where A_1 is a square matrix of order $m \times m$ which may, without loss of generality, be assumed to be non-singular. Similarly, $B = [B_1\ B_2]$ where B_1 is square. Let then P be equal to

$$\begin{bmatrix} A_1 & 0 \\ A_2 & I_{n-m} \end{bmatrix}$$

and hence P^{-1} equal to

$$\begin{bmatrix} A_1^{-1} & 0 \\ -A_2A_1^{-1} & I_{n-m} \end{bmatrix}.$$

We remember that $M = I_n - AB = \begin{bmatrix} I_m - A_1B_1 & -A_1B_2 \\ -A_2B_1 & I_{n-m} - A_2B_2 \end{bmatrix}$

and find $$P^{-1}MP = \begin{bmatrix} I_m - BA & -B_2 \\ 0 & I_{n-m} \end{bmatrix}.$$

Since the lower right-hand corner is not zero, the rank of this matrix cannot be lower than $n - m$, as was to be proved. It also follows: *For M to have precisely rank $n - m$ it is necessary and sufficient that*

$$I_m - BA = 0. \tag{5}$$

We ask now under what conditions the latent roots of $M'M$ are zero (m times) and unity ($n - m$ times). Condition (5) is clearly necessary, but not sufficient. A further necessary condition is given by the requirement that the trace (spur) of $M'M$ must be $n - m$. More explicitly, we require

$$\text{tr}\,(I_n - AB)'(I_n - AB) = \text{tr}\, I_n - 2\,\text{tr}\, AB + \text{tr}\,(AB)'(AB) = n - m. \tag{6}$$

The first term equals n and the second equals $2m$ in view of (5). Thus (6) reduces to

$$\text{tr}\,(AB)'(AB) = m. \tag{7}$$

We can now show that (5) *and* (7) *are sufficient for the discriminant to have as its latent roots m times zero and $n - m$ times unity*, by the following argument.

It is known that the least square solution, i.e. that matrix $\hat{B}$ which makes (3) a minimum, has the latent roots just mentioned (Ref. 3). $\hat{B}$ is, in fact, the solution of

$$A'y = A'Ac = A'A\hat{B}y.$$

Since this is an identity, we have

$$\hat{B} = (A'A)^{-1}A' \tag{8}$$

and it will be seen at once that $\hat{B}$ satisfies (5) and (7). If we can show that (5) and (7) admit only one solution, then it must be identical with $\hat{B}$ and hence (5) and (7) are also sufficient to produce the required latent roots. We have therefore to show that

(5) *and* (7) *have only one real solution.*

Let one solution be $\hat{B}$ and assume that $\hat{B} + D$ is also a solution. It follows from (5) that

$$DA = 0. \tag{9}$$

Furthermore, (7) requires that

$$\text{tr}\,[A(\hat{B} + D)]'\,[A(\hat{B} + D)]$$
$$= \text{tr}\,(A\hat{B})'(A\hat{B}) + 2\,\text{tr}\,(AD)'(A\hat{B}) + \text{tr}\,(AD)'(AD) = m.$$

The first term equals m because of (7). The second term is $2\,\text{tr}\,D'A'AB$ which is equal to $2\,\text{tr}\,D'A'$ by virtue of (8), and hence zero by virtue of (9). There remains as a condition $\text{tr}\,(AD)'(AD) = 0$. It follows that $AD = 0$ and hence $D = 0$. This completes the proof.

The theorem just proved can be expressed by saying: there is only one choice of B which leads to a quadratic form having a chi-squared distribution with n—m degrees of freedom.

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On a Set of Conform-Invariant Equations of the Gravitational Field

By H. A. Buchdahl

(*Received 20th July* 1949. *Read 4th November* 1949.)

1. Eddington[1] has considered equations of the gravitational field in empty space which are of the fourth differential order, viz. the sets of equations which express the vanishing of the Hamiltonian derivatives of certain fundamental invariants. The author has shown[2] that a wide class of such equations are satisfied by any solution of the equations

$$G_{\mu\nu} = \lambda g_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ and $g_{\mu\nu}$ are the components of the Ricci tensor and the metrical tensor respectively, whilst λ is an *arbitrary* constant. For a V_4 this applies in particular when the invariant referred to above is chosen from the set

$$\left.\begin{aligned} K_1 &= G^2 \,(= G^{\mu}_{\mu} G^{\nu}_{\nu}) \\ K_2 &= G_{\mu\nu} G^{\mu\nu} \\ K_3 &= B_{\mu\nu\sigma\rho} B^{\mu\nu\sigma\rho}, \end{aligned}\right\} \tag{1.2}$$

where $B_{\mu\nu\sigma\rho}$ is the covariant curvature tensor. K_3 has been included since, according to a result due to Lanczos[3], its Hamiltonian derivative $P_3{}^{\mu\nu}$ is a linear combination of $P_1{}^{\mu\nu}$ and $P_2{}^{\mu\nu}$, i.e. of the Hamiltonian derivatives of K_1 and K_2. In fact

$$P_3{}^{\mu\nu} = 4P_2{}^{\mu\nu} - P_1{}^{\mu\nu}. \tag{1.3}$$

It appears therefore that the most general invariant which will give rise to quasi-linear fourth order equations may be taken to be

$$aK_1 + bK_2, \tag{1.4}$$

where a and b are constants.

The question of the *general* solution of such equations seems to be as yet unsolved, even in the case of static spherically symmetric fields, which despite its relative simplicity presents great difficulties. In the present paper we shall be concerned with a special case of (1.4), viz. with the invariant

$$K = 3K_2 - K_1. \tag{1.5}$$

(This invariant is also mentioned in a different context by Gregory[4].) We shall show that if $P^{\mu\nu}$ is the Hamiltonian derivative of K, then *the equations*

$$P^{\mu\nu} = 0 \tag{1.6}$$

possess as a solution every line element representing a space conformal to an Einstein space.

Furthermore we shall show that the *general* solution of (1.6) in the case of the static spherically symmetric field may be written

$$ds^2 = \psi(r)\,[-\gamma^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 + \gamma\, dt^2], \tag{1.7}$$

where $\gamma = 1 - 2m/r - \lambda r^2/3$ (m, λ arbitrary constants), and $\psi(r)$ is an *arbitrary* function of r. We are of course dealing with a V_4 throughout.

2. Let $C_{\mu\nu\sigma\rho}$ be the conformal curvature tensor[5]

$$C_{\mu\nu\sigma\rho} = B_{\mu\nu\sigma\rho} - 2g_{[\mu[\nu}(G_{\sigma]\rho]} - \tfrac{1}{6}g_{\sigma]\rho]}G). \tag{2.1}$$

(For the meaning of brackets enclosing indices, *vide* Schouten[6].) Consider the invariant

$$\overline{K} = C_{\mu\nu\sigma\rho}\,C^{\mu\nu\sigma\rho}. \tag{2.2}$$

Using (2.1) it is not difficult to show that we may write

$$\overline{K} = \tfrac{1}{3}K_1 - 2K_2 + K_3. \tag{2.3}$$

Now let

$$L = K_1 - 4K_2 + K_3. \tag{2.4}$$

Inserting this in (2.3) we obtain

$$\overline{K} = L - \tfrac{2}{3}K_1 + 2K_2. \tag{2.5}$$

In virtue of (1.3) the Hamiltonian derivative of L vanishes identically. Accordingly we simply consider the invariant K as given by (1.5). The Hamiltonian derivative $P^{\mu\nu}$ of K will then be the same as that of $\overline{K}$ except for a trivial numerical factor.

3. Consider the integral

$$J = \int \overline{K}\sqrt{-g}\,d\tau, \tag{3.1}$$

where $\overline{K}$ and g are formed with respect to a metrical tensor $g_{\mu\nu}$. In a conformal transformation in which the $g_{\mu\nu}$ are replaced by $\sigma g_{\mu\nu}$, where σ is an arbitrary function of the coordinates, $\overline{K}$ becomes multiplied by σ^{-2} and $\sqrt{-g}$ by σ^2, (in a V_4). Now $\overline{P}^{\mu\nu}$ is defined by the equation

$$\delta J = \int \overline{P}^{\mu\nu}\,\delta g_{\mu\nu}\sqrt{-g}\,d\tau, \tag{3.2}$$

where variations vanish on the boundary of the region of integration. It follows that in a conformal transformation the $\bar{P}^{\mu\nu}$ merely become multiplied by σ^{-3}. The set of equations

$$\bar{P}^{\mu\nu} \equiv P^{\mu\nu} = 0 \tag{3.3}$$

therefore transforms into itself, and we accordingly say that it is *conform-invariant*. (Strictly speaking $P^{\mu\nu}$ itself is "conform-covariant".) Obviously therefore, if $\bar{g}_{\mu\nu}$ is a particular solution of (3.3), then the product of $\bar{g}_{\mu\nu}$ with an arbitrary (sufficiently often differentiable) function of the coordinates is a more general solution. Making use of the known results stated in §1 we therefore have that if $\bar{g}_{\mu\nu}$ is the metrical tensor of an Einstein space, i.e. $\bar{g}_{\mu\nu}$ satisfies the equations (1.1), and $\Lambda(x_1, x_2, x_3, x_4)$ is an arbitrary function of the coordinates, then

$$g^{*}_{\mu\nu} = \Lambda(x)\, \bar{g}_{\mu\nu} \tag{3.4}$$

satisfies the set of equations (3.3); which proves our first assertion.

4. Although it is not essential for our purpose it may be of interest to write down the explicit form of $P^{\mu\nu}$. In fact, using some results due to the author[7], we find without difficulty that

$$\tfrac{1}{2}P^{\mu\nu} = 2C^{\mu\nu\sigma\rho}{}_{;\,\sigma\rho} - C^{\mu\nu\sigma\rho}\, G_{\sigma\rho}. \tag{4.1}$$

(4.1) may also be written

$$P^{\mu\nu} = S^{\mu\nu} - \tfrac{1}{4}g^{\mu\nu}\, S, \tag{4.2}$$

where $S^{\mu\nu} = 3\,\Box\, G^{\mu\nu} - G^{;\mu\nu} + 2GG^{\mu\nu} - 6B^{\mu\nu\sigma\rho}\, G_{\sigma\rho}$, $(S = S^{\mu}_{\mu})$. (4.3)

By (4.2) the spur of $P^{\mu\nu}$ vanishes identically. [This is a general property of the Hamiltonian derivatives of fundamental invariants K which are such that the corresponding scalar-densities $K\sqrt{-g}$ are conform-invariant. This is easily proved by considering the special variation

$$\delta g_{\mu\nu} = \epsilon g_{\mu\nu} \tag{4.4}$$

in the equation of definition of Hamiltonian derivatives (cf. (3.2)), where ϵ is an arbitrary infinitesimal function of the coordinates, vanishing on the boundary of the region of integration.]

5. We now come to the case of static spherically symmetric solutions of (3.3). If we disregard trivial arbitrary constants, the only Einstein spaces having the required property are[8].

$$\left.\begin{aligned} ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \equiv -\gamma^{-1}\, d\rho^2 - \rho^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) + \gamma dt^2, \\ \text{where } \gamma = 1 - 2m/\rho - \lambda\rho^2/3, \end{aligned}\right\} \tag{5.1}$$

m and λ being constants of integration. (We consider 'different' solutions obtainable from one another by coordinate transformations as constituting the same solution.)

By a suitable choice of coordinates every spherically symmetric static line element may be brought into the form

$$ds^2 = -e^{\mu(\rho)}\,d\rho^2 - \rho^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) + e^{\nu(\rho)}\,dt^2. \tag{5.2}$$

In the present case we may further simplify (5.2) by carrying out first a conformal transformation in which the right-hand side of (5.2) is multiplied by $e^{-\nu}$ throughout, followed by a coordinate transformation $\rho = \rho(r)$ such that $\rho^2 \exp(-\nu(\rho)) = r^2$. We need then only consider line elements of the form

$$ds^2 = -(rw)^{-2}\,dr^2 - r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) + dt^2. \tag{5.3}$$

It is not difficult to confirm that the components of the curvature tensor do not contain the second derivatives of w. Consequently the same is true of K, and in fact we find

$$K = 2(rww' + r^{-2})^2. \tag{5.4}$$

The condition that the Hamiltonian derivative of K vanish then yields the *second* order differential equation

$$\left(-\tfrac{1}{2}\frac{hK}{hw}\,r^4w = \right)\ 2r^6w^3w'' + 6r^5w^3w' + r^6w^2w'^2 + 1 = 0. \tag{5.5}$$

If $r = \tau^{-1/2}$ and dots denote differentiation with respect to τ, (5.5) becomes

$$8w^3\ddot{w} + 4w^2\dot{w}^2 + 1 = 0, \tag{5.6}$$

which is easily integrated, its solution being

$$w^2 + 4mw^3 = \sigma - 27m^2\sigma^2, \tag{5.7}$$

where we have written $\sigma = \tau - \lambda/3$, and m, λ are constants of integration. In terms of a parameter ρ (5.7) may be given the equivalent form

$$\sigma\rho^3 - \rho + 2m = 0 \tag{5.81}$$

$$w = \rho^{-1} - 3m\rho^{-2}. \tag{5.82}$$

But if we now apply the transformations described above to the line element (5.1) then the relation between ρ and r is just that given by (5.81); and it may be confirmed that (5.82) in fact correctly represents the resulting function $w(r)$. It follows at once that all static spherically symmetric solutions of (3.3) can be written in the form (1.7); which was to be proved.

It may be noted that just the set of equations (3.3) is obtained in Weyl's theory[10] if we attempt to set up field equations by choosing

$$\delta\int \overline{C}_{\mu\nu\sigma\rho}\,\overline{C}^{\mu\nu\sigma\rho}\,\sqrt{-\overline{g}}\,d\tau = 0 \tag{5.9}$$

as the determining gauge-invariant action principle. This is of course not surprising since Weyl's conformal curvature tensor [11] $\overline{C}_{\mu\nu\rho}$ does not involve the "electromagnetic potentials" at all. But it is interesting to observe that in this case we can at least obtain convergent solutions of the field equations (cf. Bergmann [12]). On the other hand the "unity" of gravitation and electromagnetism is then of an even more dubious kind.

In conclusion I wish to express my thanks to a referee who pointed out a missing link in the argument of the last section in the original draft of this paper.

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Note added in proof:

In the transformation of coordinates following (5.2) we have tacitly assumed that $\nu(\rho) \neq 2 \log \rho$. If this condition is not satisfied we need only consider a line element of the form

$$ds^2 = -e^{\lambda(\rho)} d\rho^2 - (d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2.$$

Then $K = 2$ and the Hamiltonian derivative of K is 1. The result of §5 therefore remains valid.

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On a Certain Matrix Product with Specified Latent Roots

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(*Received* 19*th July* 1949. *Read* 4*th November* 1949.)

1. Vajda's paper [1] in this volume has suggested to the author the following problem:

Let A be a $n \times m$ matrix, $m \leqq n$, let B be a $m \times n$ matrix, and let $M = I - AB$, where I is the unit matrix of order n. Given A, to find B such that of the n latent roots of $M'M$, k are unity, and the remaining $n - k$ are zero.

The case considered by Vajda is that in which both A and $I - M$ are of maximum rank m; the present note takes the solution a little further without, however, answering the problem completely.

2. Consider first the more stringent requirement

$$M'M = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

where I_k is the unit matrix of order k, defining by its position a mode of partitioning of the $n \times n$ matrix $M'M$. In the same mode of partitioning let

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}.$$

We then have

$$m_1' m_1 + m_3' m_3 = I_k, \tag{2}$$

$$m_2' m_2 + m_4' m_4 = 0. \tag{3}$$

If, as we shall assume, M is real, (3) gives $m_2 = 0$, $m_4 = 0$, since both $m_2' m_2$ and $m_4' m_4$ are non-negative definite matrices (ref. (1), p. 97, ex. 2). Hence

$$M = \begin{bmatrix} m_1 & 0 \\ m_3 & 0 \end{bmatrix}$$

and $I - M$ is at least of rank $n - k$; but since it is at most of the rank of A, which we shall denote by r, where $r \leqq m$, we have $n - k \leqq r$, or $k \geqq n - r$.

---

[1] Ref. (2).

3. Let $k-(n-r)=s$, where s is one of the integers 0, 1, ..., r, and put

$$M=\begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & 0 \end{bmatrix}, \qquad A=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

where m_{11} is a $(n-r)\times(n_1'-r)$ matrix, m_{22} is a $s\times s$ matrix, while the partitioning of A, to accord with that of M, is such as to let a_1, a_2 and a_3 have $n-r$, s and $r-s$ rows respectively. Without loss of generality we may assume that r linearly independent rows of A are formed by those of a_2 and a_3. The following theorem holds:

For the existence of a solution B it is necessary and sufficient for the rows of a_1 to be independent of those of a_3.

The proof is an application of a general theorem on linear equations by which there exists a solution B if and only if the linear relations between the rows of A hold also for the rows of $I-M$.

Assume first that the rows of a_1 depend linearly on those of a_2 and a_3, so that there is a relation of the form $a_1=Ca_2+Da_3$, where C and D are matrices. Considering the last $r-s$ columns of $I-M$ we must then have $0=C\times 0+D\times I$, or $D=0$. Hence $a_1=Ca_2$, which establishes the necessity of the condition. It is also sufficient, for when it is satisfied we can determine m_{11} and m_{12} from the condition

$$[I_{n-r}-m_{11}, \qquad -m_{12}]=C[-m_{21}, \qquad I_s-m_{22}],$$

where I_{n-r}, I_s are unit matrices of the orders of their suffixes.

In order that condition (2) should also be satisfied we must further have

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}'\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}+[m_{31} \quad m_{32}]'[m_{31} \quad m_{32}]=\begin{bmatrix} I_{n-r} & 0 \\ 0 & I_s \end{bmatrix}.$$

The preceding requirements can all be seen to be satisfied if we take

$$M=\begin{bmatrix} I_{n-r} & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

B is then obtained by solving the system of equations

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}B=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{r-s} \end{bmatrix}.$$

This completes the demonstration of the sufficiency of the condition enunciated in the theorem.

4. In this section we shall discuss the particular case where $s = 0$ or $n - r = k$. Then a_2 has no rows and $a_1 = 0$. Hence we have to solve

$$\begin{bmatrix} 0 \\ a_3 \end{bmatrix} B = \begin{bmatrix} I_{n-r} - m_1 & 0 \\ -m_3 & I_r \end{bmatrix};$$

and, by considering the rank of the matrix formed by adjoining to A columns of $I - M$, we see that $I_{n-r} - m_1 = 0$. By (2) it then follows that $m_3 = 0$, and we solve for B from the system of equations $a_3 B = [0\ I_r]$. In this case the determination (4) of M is the only possible one.

We now return to the original problem, proposed in § 1, where M is a matrix such that $M'M$ has k latent roots which are unity, and $n - k$ which are zero. In this case there exists an orthogonal matrix H such that, in analogy with (1),

$$H'M'MH = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}. \tag{5}$$

Since $H'H = I$ the equations for solution, $I - AB = M'M$, are equivalent to $I - H'ABH = H'M'MH$, which are of the form already solved except that where previously we had A and B we now have $H'A$ and BH. But by a previous result a solution BH exists if and only if there is a relation of the form

$$H'A = \begin{bmatrix} 0 \\ a_3 \end{bmatrix}. \tag{6}$$

If we partition H by writing $H = [h_1\ h_2]$, when h_2 has r columns, we obtain from (6) the relations

$$h_2'A = a_3 \tag{7}$$

and

$$A = H \begin{bmatrix} 0 \\ a_3 \end{bmatrix} = h_2 a_3. \tag{8}$$

Now as a_3 is of rank r the matrix product $a_3 a_3'$, which is of order $r \times r$, is non-singular and so possesses an inverse (ref. (1), p. 97, ex. 3).

Hence we may write $Aa_3' = h_2 a_3 a_3'$, or $h_2 = Aa_3'(a_3 a_3')^{-1}$. On the other hand, A being of rank r, let A_1 be the matrix formed by some r linearly independent columns contained in A. Then there exists a matrix K such that $A = A_1 K$. Hence

$$h_2 = A_1 K a_3'(a_3 a_3')^{-1} = A_1 D, \tag{9}$$

say. Since h_2 and A_1 are both of rank r, the $r \times r$ matrix $D = K a_2' (a_2 a_3')^{-1}$ is also of rank r and hence possesses an inverse. Combining (7) and (9) we obtain

$$a_3 = D' A_1' A. \tag{10}$$

Now B is found from the relation $H'ABH = I - H'M'MH$, which, in virtue of (5) and (6), reduces to

$$\begin{bmatrix} 0 \\ a_3 \end{bmatrix} BH = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}$$

or $a_3 BH = [0\ I_r]$, or $a_3 B = [0\ I_r] H' = h_2'$. Substituting in this from (9) and (10) we have

$$D' A_1' AB = D' A_1'.$$

Since D, and so D', possesses an inverse this reduces finally to

$$A_1' AB = A_1'. \tag{11}$$

This is an assemblage of n systems of r equations each and m ($\geqq r$) unknowns each; and can therefore be solved. Although A_1', consisting of any r linearly independent rows of A', is not determinate, equations (11) are not therefore indeterminate; for any two different determinations of A_1' can be transformed into each other by premultiplying by a non-singular matrix of order r; and similarly the corresponding versions of equation (11) can be transformed into each other by premultiplying its two sides by the same matrix. This premultiplication leaves the value of B unchanged.

If $m = r$, or A has maximum rank, A_1 is identical with A, and the solution for B is then unique, being

$$B = (A'A)^{-1} A'.$$

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On the Φ-Invariant of Two Quadrics

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1. *Introduction.*

The geometry associated with the five invariants of two quadrics is well known.[1] There appears to be an omission, however, in the treatment of the Φ-invariant. Salmon gives the vanishing of Φ merely as a *necessary* condition for the possibility of the construction of a tetrahedron self-conjugate for one of the quadrics and having its six edges tangential to the other.[2] Sommerville proves its *sufficiency*, and shows that this condition is poristic, giving rise to two systems of ∞^1 tetrahedra of the requisite type.[3] No investigation appears to have been made of the locus of the vertices of the ∞^1 tetrahedra of each system and the dual problem regarding the nature of the developable surface arising from their ∞^1 faces.

The purpose of this paper is to derive the equations of these constructs, showing the curve to be of order eight, and the developable to be of class eight. The treatment throughout is based on the Clebsch-Aronhold symbolism,[4] viz.,

$$
\begin{array}{llll}
f \equiv a_x^2, & f' \equiv b_x^2. & & \\
\Sigma \equiv u_a^2, & \Sigma' \equiv u_\beta^2, & & (\alpha \equiv aa'a'',\ \beta \equiv bb'b''). \\
\Pi \equiv (Ap)^2, & \Pi' \equiv (Bp)^2, & & (A \equiv aa',\quad B \equiv bb'). \\
k \equiv (A\beta x)^2, & k' \equiv (B\alpha x)^2, & [(A\beta x) \equiv \overset{\bullet}{a}_\beta \overset{\bullet}{a}'_x & \equiv \bar{a}_\beta \bar{a}'_x]. \\
\chi \equiv (Abu)^2, & \chi' \equiv (Bau)^2. & &
\end{array}
\tag{1.1}
$$

The five invariants are then represented as follows:—

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[1] (1) Salmon, *A Treatise on the Analytic Geometry of Three Dimensions* (revised by Rogers), Ch. IX.

(2) Sommerville, *Analytical Geometry of Three Dimensions*, Ch. XV.

(3) H. W. Turnbull, "Some Geometrical Interpretations of the Concomitants of Two Quadrics," *Proc. Cambridge Phil. Soc.*, XIX (1919), 196-206.

[2] (1) p. 204, § 201.

[3] (2) p. 312, § 15.22.

[4] (3) p. 198, §§ 4, 5.

$$(1.2) \qquad \Delta \equiv a_a^2,\ \Theta \equiv b_a^2,\ \Phi \equiv (AB)^2,\ \Theta' \equiv a_\beta^2,\ \Delta' \equiv b_\beta^2.$$

In the following account it will be proved that, when Φ vanishes, the developable containing the faces of one system of the above-mentioned tetrahedra is formed by the ∞^1 planes common to the quadric and quartic envelopes

$$\chi = 0, \qquad \Delta\Sigma'^2 - 16\,\Theta'\Sigma\Sigma' + 12\,\Delta'\Sigma\chi' = 0;$$

while the locus of the ∞^1 vertices is the curve of intersection of the quadric and quartic surfaces

$$k = 0, \qquad \Delta f'^2 - 16\,\Theta f f' + 8 f k' = 0.$$

The equations of the developable and curve associated with the second system of tetrahedra will also be given in similar form.

2. *The vanishing of Φ as a necessary condition.*

Let $PQRS$ be a tetrahedron self-congugate for f' and having its six edges touching f. If u represents the plane QRS then the conic envelope of lines in u belonging to Π is given by

$$(Auv)^2 = 0 \ (v \text{ current}).$$

The triangle QRS therefore circumscribes the conic $(Auv)^2$ and is self-conjugate for the section of f' by u. That is, $(Auv)^2$ is inpolar to f', and so u satisfies the equation

$$(Abu)^2 = 0.$$

Thus, the plane QRS touches the quadric envelope χ. Similarly, all four faces of the tetrahedron $PQRS$ touch χ, and since the tetrahedron circumscribes χ and is self-conjugate for f', χ is inpolar to f', and so $(Abb')^2 = 0$.

Therefore, the vanishing of $(AB)^2$ is a *necessary* condition for the possibility of the above construction.

For convenience such a construct will now be termed a Φ-*tetrahedron* of the two quadrics.

3. *Considerations of duality.*

It can easily be shown that the vertices P, Q, R and S of a Φ-tetrahedron lie on the quadric k, the vanishing of Φ being the condition for k and χ to be polar reciprocal quadrics with respect to f'. For the polar reciprocal of χ with respect to f' is given by

$$b'_x\,(Abb')\;(Abb'')\;b''_x = 0,$$

i.e. $$\tfrac{1}{2}\dot{b}'_x\,(Abb')\;(A\dot{b}b'')\;b''_x = 0,$$

which by determinantal permutations gives

$$\tfrac{1}{2}\,(Abb')^2\,b''^2_x - \tfrac{1}{2}\dot{a}'_x\,(Abb')\,(\dot{a}b''bb')b''_x = 0,$$

reducing to $$\tfrac{1}{2}\,(AB)^2 b'^2_x - \tfrac{1}{6}\,(A\beta x)^2 = 0,$$

$$3\Phi f' - k = 0, \tag{3.1}$$

and for Φ zero this gives $k = 0$.

Thus, each Φ-tetrahedron has its vertices on k, its faces touching χ, its edges belonging to Π, and is self-conjugate for Π′. The vanishing of Φ is obviously also a necessary condition for a Φ-tetrahedron to exist with the same relationships to k', χ', Π′ and Π.

4. *A further property of k and χ.*

Before the sufficiency of the condition is considered, the following additional property associated with k and χ will be established as it is required at a later stage:

$$\begin{aligned}
(A\beta.\ Bu)^2 &= [\dot{a}_\beta\,(\dot{a}'Bu)]^2 \\
&= [\dot{b}_\beta\,(A\dot{b}'u)]^2 + 2\,(AB)\,(A\dot{b}'u)\,\dot{b}_\beta u_\beta, \qquad \text{since } (AB)^2 = 0 \\
&= 2b^2_\beta\,(Ab'u)^2 - 2b_\beta\,(Ab'u)\,b'_\beta\,(Abu) + 4(AB)\,(Ab'u)\,b_\beta u_\beta \\
&= 2b^2_\beta\,(Ab'u)^2 - \tfrac{1}{2}b'^2_\beta\,(Abu)^2 - (Ab'u)^2 b^2_\beta \\
&= \tfrac{1}{2}\Delta'\chi.
\end{aligned}$$

Hence, if the plane u belongs to χ (when Φ is zero), it also satisfies

$$(A\beta.\ Bu)^2 = 0. \tag{4.1}$$

Therefore, by the Clebsch Transference Principle every plane belonging to χ cuts k in a conic which is outpolar to the conic section of f' by u.

5. *Investigation of the sufficiency of the condition* $\Phi = 0$.

In the following it will now be assumed that the quadrics f and f' are subject only to the condition that the invariant Φ vanishes. A plane u will cut f in the conic section given by

$$(Auv)^2 = 0.$$

Let the plane u belong to χ. Then the polar reciprocal cone of $(Auv)^2$ with respect to f' is obtained from the equation

$$b_x\,(Abu)\,(Ab'u)\,b'_x = 0,$$

where
$$\begin{aligned} b_x\,(Abu)\,(Ab'u)\,b'_x &= \dot{b}_x\,(Abu)\,(A\dot{b}'u)\,b'_x, \quad \text{since } (Abu)^2 = 0 \\ &= -\,(uABx)\,(uAb)\,b'_x \\ &= -\tfrac{1}{2}\,(uABx)^2. \end{aligned} \tag{i}$$

Let the point y be the pole of u with respect to f'. From (3.1) y lies on the quadric k, and from the foregoing construction is also the vertex of the polar reciprocal cone of the conic $(Auv)^2$ with respect to f'.

Thus,
$$\begin{aligned} (uBAx)^2 &= b_y\,(bBAx)\,(b'BAx)\,b'_y \\ &= 2b_y\,(bBAx)\,(b'Ba)\,a'_x b'_y \\ &= \tfrac{2}{3}\dot{b}_y\,(bBAx)\,(b'\dot{b}_1\dot{b}_2 a)\,a'_x b'_y, \text{ where } B = b_1 b_2 \\ &= \tfrac{2}{3}\dot{b}'_y\,(bBAx)\,(bB\dot{a})\,a'_x b'_y \\ &= \tfrac{2}{3}\dot{b}'_y a_\beta a'_x \dot{a}_\beta a'_x b'_y \\ &= \tfrac{2}{3}\,(A\beta x)\,a_\beta a'_x b'^2_y - \tfrac{2}{3}\,(A\beta x)\,a_y b'_\beta a'_x b'_y \\ &= \tfrac{1}{3}\,(A\beta x)^2 b^2_y - \tfrac{1}{6} b'^2_\beta \dot{a}'_x a_y a'_x \dot{a}_y \\ &= \tfrac{1}{3}\,(A\beta x)^2 b^2_y - \tfrac{1}{12} b^2_\beta (Ayx)^2. \end{aligned} \tag{ii}$$

Also
$$\begin{aligned} (uBAx)\,(BA) &= b_y\,(bBAx)\,(BA) \\ &= -\,2b_y\,(abB)\,a'_x\,(aa'b_1 b_2) \\ &= -\,\tfrac{2}{3}\dot{b}_y\,(abB)\,a'_x\,(aa'\dot{b}_1\dot{b}_2) \\ &= -\,\tfrac{2}{3}\dot{a}'_y a_\beta a'_x \dot{a}_\beta \\ &= -\,\tfrac{1}{3}\,(A\beta y)\,(A\beta x). \end{aligned} \tag{iii}$$

By combining (i), (ii) and (iii) and using the identity

$$(uABx) \equiv -\,(uBAx) - (BA)\,u_x,$$

we have
$$\begin{aligned} (uABx)^2 &\equiv (uBAx)^2 + 2\,(uB\dot{A}\dot{x})\,(BA)\,u_x, \quad \text{since } \Phi = 0 \\ &\equiv \tfrac{1}{3}\,(A\beta x)^2 b^2_y - \tfrac{1}{12} b^2_\beta (Ayx)^2 - \tfrac{2}{3}\,(A\beta y)\,(A\beta x)\,b_y b_x. \end{aligned}$$

Thus, the polar reciprocal cone of $(Auv)^2$ with respect to f', where u is a plane of χ and Φ vanishes, is given by the equation

$$(5.1) \qquad 4b_y^2(A\beta x)^2 - b_\beta^2(Ayx)^2 - 8(A\beta y)(A\beta x)b_y b_x = 0 \quad (x \text{ current}).$$

To construct a Φ-tetrahedron let us take, as in §2, a triangle QRS self-conjugate for the section of f' by u and circumscribing $(Auv)^2$, where u is a plane touching χ, and $P(y)$ is the pole of u with respect to f'. In order to complete the construction it will be necessary to choose u and the triangle QRS so that PQ, PR and PS are generators common to the cones $(Ayx)^2$ and the polar reciprocal of $(Auv)^2$ with respect to f'. This condition is essential since PQ, PR and PS must be lines of Π and also the respective polar lines of RS, SQ and QR, which touch $(Auv)^2$.

From (5.1) it is seen that the points common to the quadric k, the polar reciprocal cone of $(Auv)^2$ with respect to f' and the plane u (viz. $b_y b_x = 0$) also lie on the cone $(Ayx)^2$. Thus, if a plane u of χ can be obtained containing a triangle QRS whose sides touch $(Auv)^2$ and whose vertices lie on the section of k by u, then since the triangle QRS is self-conjugate for f' and $P(y)$ is the pole of u with respect to f', the lines PQ, PR and PS will be generators of the polar reciprocal cone of $(Auv)^2$ with respect to f' and also generators of $(Ayx)^2$.

The problem of determining a Φ-tetrahedron is thus reduced from a problem in [3] to one in [2], viz. to obtain the planes u touching χ, which have a triangle QRS circumscribing the conic $(Auv)^2$, inscribed in the section of k by u and self-conjugate for the section of f' by u.

6. *Lemma concerning a configuration of three conics in a plane.*

To solve the problem outlined at the end of §5 the following three ternary forms and their duals will be considered:—

$$f_1 \equiv a_x^2,\ \Sigma_1 \equiv u_\alpha^2;\ f_2 \equiv b_x^2,\ \Sigma_2 \equiv u_\beta^2;\ f_3 \equiv c_x^2,\ \Sigma_3 \equiv u_\gamma^2.$$

It is to be noted that the symbols a, α; b, β; c, γ here used are in no wise related to the corresponding symbols used in connection with the quaternary forms.

The following identities connecting the non-symbolical and symbolical forms of the concomitants of two conics will be found necessary, and can easily be verified:

$$f \equiv \sum_{i,j=1}^{3} a_{ij}x_i x_j \equiv a_x^2,$$

$$\Sigma \equiv \sum_{i,j=1}^{3} A_{ij}u_i u_j \equiv \tfrac{1}{2}u_\alpha^2,$$

(6.1) $$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv \tfrac{1}{6} a_\alpha^2,$$

$$\Theta \equiv \sum_{i,j=1}^{3} b_{ij} A_{ij} \equiv \tfrac{1}{2} b_\alpha^2 .$$

Necessary and sufficient conditions for the existence of a triangle circumscribed about the conic f_1, inscribed in f_2 and self-conjugate for f_3 will now be determined.

Obviously, the following conditions will be necessary:—

(6.2) $b_\gamma^2 = 0$, since f_2 is outpolar to Σ_3,

(6.3) $c_\alpha^2 = 0$, since Σ_1 is inpolar to f_3.

There is a third necessary condition, however, arising from the fact that a triangle is inscribed in f_2 and circumscribed about f_1. In non-symbolical form the necessary (and sufficient) condition for this last construction is $\Theta^2 = 4\Delta\Theta'$.[1]

From (6.1) the symbolical form of this third necessary condition is therefore

(6.4) $$b_\alpha^2 b'^2_\alpha = \tfrac{1}{3} a_\alpha^2 a'^2_\beta ,$$

which by the interchange of equivalent symbols reduces to the slightly modified form

(6.4(a)) $$b_\alpha^2 b'^2_\alpha = (aa'\beta)^2 .$$

It can easily be shown that the condition (6.4(a)) also covers the case where Σ_1 degenerates to a point pair.

It will now be proved that conditions (6.2), (6.3) and (6.4) are also sufficient for the possibility of the above construction, and yield a unique triangle satisfying the conditions.

The configuration now to be considered consists of the conic envelope Σ_1, and conic loci f_2 and f_3 subject to conditions (6.2), (6.3) and (6.4(a)). Since the envelope Σ_1 is taken as fundamental, its point form becomes $S_1 \equiv (aa'x)^2$, where $(aa'x)^2 \equiv \tfrac{1}{3} a_\alpha^2 f_1$, and $(aa'\beta)^2$ is the Θ'-invariant of S_1 and Σ_2. Also, since b_γ^2 is zero, the polar reciprocal of f_2 with respect to f_3 is their Φ-conic,[2] viz.

$$\Phi_{23} \equiv (bcu)^2 = 0.$$

[1] (4) Sommerville, *Analytical Conics*, p. 279, § 23.

[2] (4) p. 286, Ex. 2., but the result can easily be proved symbolically.

There are four tangents common to the conic envelopes Σ_1 and Φ_{23}, and these determine the range of conic envelopes

$$\Sigma_\lambda \equiv (bcu)^2 + \lambda u_a^2 = 0.$$

It can be shown by applying the Clebsch Transference Principle to the binary domain that the point equation of Σ_λ is

$$S_\lambda \equiv (bc.\, b'c'.\, x)^2 + 2\lambda\,(bc.\, a.\, x)^2 + \lambda^2\,(aa'x)^2 = 0.$$

The Θ'-invariant of S_λ and Σ_2 is therefore

$$(bc.\, b'c'.\, \beta)^2 + 2\lambda\,(bc.\, a.\, \beta)^2 + \lambda^2\,(aa'\beta)^2,$$

and so for a triangle to be inscribed in f_2 and circumscribed about Σ_λ the following necessary and sufficient condition of type (6.4(a)) will have to be proved:—

$$(6.5)\quad [(bcb')^2 + \lambda b_a^2]\,[(b''c'b''')^2 + \lambda b''^2_a] \equiv (bc.\, b'c'.\, \beta)^2 + 2\lambda\,(bc.\, a.\, \beta)^2 + \lambda^2\,(aa'\beta)^2,$$

where b, b''; c, c'; a, a' etc. are equivalent pairs of symbols.

The left-hand side of the above equation reduces to

$$(c_\beta^2 + \lambda b_a^2)\,(c'^2_\beta + \lambda b'^2_a) = c_\beta^2 c'^2_\beta + 2\lambda b_a^2 c_\beta^2 + \lambda^2 b_a^2 b'^2_a,$$

while the right-hand side becomes

$$\begin{aligned}
&[(bb'c')\,c_\beta - (cb'c')\,b_\beta]^2 + 2\lambda\,[b_a c_\beta - c_a b_\beta]^2 + \tfrac{4}{3}\lambda^2 a_a^2 a'^2_\beta \\
&= c_\beta^2 c'^2_\beta + b_\beta^2 b'^2_\gamma - 2c_\beta b_\beta\,(bb'c')\,(cb'c') + 2\lambda\,(b_a^2 c_\beta^2 + c_a^2 b_\beta^2 - 2b_a c_\beta b_\beta c_a) \\
&\qquad + \tfrac{4}{3}\lambda^2 a_a^2 a'^2_\beta \\
&= c_\beta^2 c'^2_\beta + b_\beta^2 b'^2_\gamma - \tfrac{2}{3} b_\beta^2 b'^2_\gamma + 2\lambda\,(b_a^2 c_\beta^2 + c_a^2 b_\beta^2 - \tfrac{2}{3} b_\beta^2 c_a^2) + \tfrac{4}{3}\lambda^2 a_a^2 a'^2_\beta \\
&= c_\beta^2 c'^2_\beta + 2\lambda b_a^2 c_\beta^2 + \tfrac{4}{3}\lambda^2 a_a^2 a'^2_\beta,
\end{aligned}$$

since relations (6.2) and (6.3) are satisfied by the three conics f_1, f_2 and f_3.

Also, from (6.4) $\qquad b_a^2 b'^2_a = \tfrac{4}{3} a_a^2 a'^2_\beta.$

Thus, (6.5) is established, and so for all values of λ an infinity of triangles can be inscribed in f_2 and circumscribed about Σ_λ.

Now let the four tangents common to Σ_1 and Φ_{23} meet in the point pairs A, A'; B, B'; and C, C', which are three degenerate conic envelopes of the range Σ_λ. By application of (6.5) to the point pair A, A' a triangle can therefore be inscribed in f_2 and circumscribed about the degenerate conic envelope whose tangent lines pass through A, A'. Thus, either A or A' must lie on f_2, and similarly, B or B' and C or C' must also lie on f_2. As it is impossible for three collinear

points to lie on a conic, there is no loss of generality in taking the points A, B and C as the three points on f_2. Therefore, the triangle ABC is inscribed in f_2 and circumscribed about Φ_{23} and Σ_1. Moreover, since Φ_{23} is the polar reciprocal of f_2 with respect to f_3 and f_2 is outpolar to f_3, this triangle is also self-conjugate for f_3.

The triangle ABC is therefore a unique triangle circumscribing Σ_1, inscribed in f_2 and self-conjugate for f_3, provided (6.2), (6.3) and (6.4) are satisfied.

7. *Application to the* Φ*-tetrahedron of two quadrics.*

In the case of the three-dimensional configuration it was shown in §5 that in order to construct a Φ-tetrahedron, when Φ vanishes, it is necessary to obtain a plane u belonging to χ, having a triangle circumscribing $(Auv)^2$ inscribed in the section of k by u, and self-conjugate for the section of f' by u.

From (4.1), provided u belongs to χ, then $(Auv)^2$ is inpolar to f', and the section of k by u is outpolar to the section of f' by u. Thus, if $(Auv)^2$ is taken to correspond to Σ_1, the section of k by u to correspond to f_2, and the section of f' by u to correspond to f_3, where Σ_1, f_2 and f_3 represent the conics dealt with in §6, then (6.2) and (6.3) are satisfied by the conic sections of f, k and f' by u. By the application of the Clebsch Transference Principle to relation (6.4) will be found the ∞^2 planes u, which are such that a triangle can be inscribed in the section of k by u, and circumscribed about the section of f by u. This condition is therefore

$$(7.1) \qquad [(A\beta.\, A'u)^2]\,[(A''\beta'.\, A'''u)^2] = \tfrac{4}{3}u_a^2(a.\, A\beta.\, A'\beta'.\, u)^2,$$

which represents a surface of class four. The ∞^1 planes common to this surface and the quadric envelope χ therefore produce conic sections which satisfy conditions (6.2), (6.3) and (6.4) simultaneously, and so constitute the developable surface of class eight whose planes are the faces of one system of Φ-tetrahedra, viz. that system whose edges touch f, and whose members are self-conjugate for f'. Dually, the ∞^1 vertices of this system will describe an octavic curve as locus.

Expressed in terms of the irreducible concomitants of two quadrics the developable is given by the equations

$$(7.2) \qquad \chi = 0 \text{ and } \Delta\Sigma'^2 - 16\Theta'\Sigma\Sigma' + 12\Delta'\Sigma\chi' = 0,$$

and the curve by the equations

$$(7.3) \qquad k = 0 \text{ and } \Delta f'^2 - 16\Theta f f' + 8fk' = 0.$$

The reduction of (7.1) and its dual to forms (7.2) and (7.3) is rather long and involved, and consequently the detailed proof is included as an Appendix.

Similarly, for the second system of Φ-tetrahedra whose edges touch f' and whose members are self-conjugate for f, the developable and curve are given respectively by the equations

(7.4) $\chi' = 0$ and $\Delta'\Sigma^2 - 16\,\Theta\Sigma\Sigma' + 12\,\Delta\Sigma'\chi = 0,$

(7.5) $k' = 0$ and $\Delta' f^2 - 16\,\Theta' f f' + 8 f' k = 0.$

8. *Conclusion.*

Certain aspects of the foregoing theory can be applied to the case where Φ is the $(AB)^2$ invariant of a quadratic complex $(Ap)^2$ and a quadric $(Bp)^2$ with further modifications where a Battaglini complex is involved.

In the case of the Φ-invariant of two general quadratic complexes, $(Ap)^2$ and $(Bp)^2$, an entirely different method of approach is necessary.

These considerations are deferred meantime.

Appendix.

The following is the detailed working involved in deriving relations (7.2)–(7.5) from equation (7.1), viz.

$$(A\beta.\ A'u)^2\,(A''\beta'.\ A''u)^2 = \tfrac{1}{3}u_\alpha^2\,(a.\ A\beta.\ A'\beta'.u)^2.$$

Firstly, $(A\beta.\ A'u)^2 = [\dot{a}_\beta\,(\dot{a}'A'u)]^2$

$$= 2a_\beta^2 u_\alpha^2 - 2a_\beta a'_\beta\,(aA'u)\,(a'A'u)$$

$$= 2a_\beta^2 u_\alpha^2 - \tfrac{2}{3}\dot{a}_\beta a'_\beta\,(aA'u)\,(a'\dot{a}_1\dot{a}_2 u), \text{ where } A' \equiv a_1 a_2,$$

$$= 2a_\beta^2 u_\alpha^2 - \tfrac{2}{3}a'^2_\beta u_\alpha^2 + \tfrac{2}{3}u_\beta a'_\beta u_\alpha a'_\alpha$$

$$= \tfrac{4}{3}a_\beta^2 u_\alpha^2 + \tfrac{1}{6}a_\alpha^2 u_\beta^2.$$

(A.1) Thus, $(A\beta.\ A'u)^2\,(A''\beta'.\ A''u)^2 = \tfrac{16}{9}a_\beta^2 a'^2_{\beta'} u_\alpha^2 u_{\alpha'}^2 + \tfrac{1}{36}a_\alpha^2 a'^2_{\alpha'} u_\beta^2 u_{\beta'}^2 + \tfrac{4}{9}a_\alpha^2 a'^2_\beta u_{\alpha'}^2 u_{\beta'}^2.$

Secondly, $(a.\ A\beta.\ A'\beta'.\ u)^2 = [\dot{a}_{1\beta_1}\bar{a}_{2\beta_2}\,(\dot{a}_1'\bar{a}_2' a u)]^2,$

where $A\beta \equiv \dot{a}_{1\beta_1}\dot{a}_1'$ and $A'\beta' \equiv \bar{a}_{2\beta_2}\bar{a}_2'.$

This reduces to

$$[a_{1\beta_1}a_{2\beta_2}(a'_1 a'_2 a u) - a'_{1\beta_1}a_{2\beta_2}(a_1 a'_2 a u) + a'_{1\beta_1}a'_{2\beta_2}(a_1 a_2 a u) - a_{1\beta_1}a'_{2\beta_2}(a'_1 a_2 a u)]^2$$

$$= 4a_\beta^2 a'^2_{\beta'} u_\alpha^2 - 2a_{1\beta_1} a'_{1\beta_1} a^2_{2\beta_2} (a'_1 a'_2 au)(a_1 a'_2 au) + 2a_{1\beta_1} a'_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a'_1 a'_2 au)(a_1 a_2 au)$$

$$- 2a^2_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a'_1 a'_2 au)(a'_1 a_2 au) - 2a'^2_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a_1 a'_2 au)(a_1 a_2 au)$$

$$+ 2a_{1\beta_1} a'_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a_1 a'_2 au)(a'_1 a_2 au) - 2a_{1\beta_1} a'_{1\beta_1} a'^2_{2\beta_2} (a_1 a_2 au)(a'_1 a_2 au).$$

In the above expression, by interchanging equivalent symbols a_1 with a_2 and a'_1 with a'_2 in the fourth term; a'_1 with a_2 and a'_2 with a_1 in the fifth term; a_1 with a'_1 in the sixth term; and a_2 with a'_2 in the seventh term, it may be shown that

$$(A.2)\quad (a.A\beta.A'\beta'.u)^2 = 4a_\beta^2 a'^2_{\beta'} u_\alpha^2 - 8a_{1\beta_1} a'_{1\beta_1} (a'_1 a'_2 au)(a_1 a'_2 au) a^2_{2\beta_2} + 4a_{1\beta_1} a'_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a'_1 a'_2 au)(a_1 a_2 au)$$

$$= 4(X - 2Y + Z),$$

where $X = a_\beta^2 a'^2_{\beta'} u_\alpha^2$,

$$Y = a_{1\beta_1} a'_{1\beta_1} (a'_1 a'_2 au)(a_1 a'_2 au) a^2_{2\beta_2}$$

$$= \tfrac{1}{3} \dot{a}_{1\beta_1} a'_{1\beta} (a'_1 \dot{a}'_2 \dot{a}u)(a_1 a'_2 au) a^2_{2\beta_2}$$

$$= \tfrac{1}{3} a'^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 - \tfrac{1}{3} u_{\beta_1} a'_{1\beta_1} a'_{1\alpha} u_\alpha a^2_{2\beta_2}$$

$$= \tfrac{1}{3} a_\beta^2 a'^2_{\beta'} u_\alpha^2 - \tfrac{1}{12} a_\alpha^2 a'^2_\beta u_\beta^2,$$

$$Z = a_{1\beta_1} a'_{1\beta_1} a_{2\beta_2} a'_{2\beta_2} (a'_1 a'_2 au)(a_1 a_2 au)$$

$$= \tfrac{1}{6} a_{1\beta_1} \dot{a}'_{1\beta_1} a_{2\beta_2} \dot{a}'_{2\beta_2} (a'_1 a'_2 au)(a_1 a_2 \dot{a}u)$$

$$= \tfrac{1}{6} [a_{1\beta_1} \dot{a}_{1\beta_1} a_{2\beta_2} \dot{a}_{2\beta_2} u_\alpha^2 - a_{1\beta_1} \dot{u}_{\beta_1} a_{2\beta_2} \dot{a}_{2\beta_2} u_\alpha a_{1\alpha} + a_{1\beta_1} \dot{u}_{\beta_1} a_{2\beta_2} \dot{a}_{1\beta_2} u_\alpha a_{2\alpha}]$$

$$= \tfrac{1}{6} [a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 - a_{1\beta_1} a_{2\beta_1} a_{2\beta_2} a_{1\beta_2} u_\alpha^2 - a_{1\beta_1} u_{\beta_1} a^2_{2\beta_2} u_\alpha a_{1\alpha}$$

$$+ a_{1\beta_1} a_{2\beta_1} a_{2\beta_2} u_\beta u_\alpha a_{1\alpha} + a_{1\beta_1} u_{\beta_1} a_{2\beta_2} a_{1\beta_2} u_\alpha a_{2\alpha} - a^2_{1\beta_1} a_{2\beta_2} u_\beta u_\alpha a_{2\alpha}].$$

The above six terms of Z are reduced as follows:—

1st term $= a_\beta^2 a'^2_{\beta'} u_\alpha^2$.

2nd term $= a_{1\beta_1} a_{2\beta_1} a_{2\beta_2} a_{1\beta_2} u_\alpha^2$

$$= a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 + a_{1\beta_1} \dot{a}_{2\beta_1} a_{2\beta_2} \dot{a}_{1\beta_2} u_\alpha^2$$

$$= a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 + (a_1 b_1 b_2 b_3)(a_2 a_1 \dot{b}_2 \dot{b}_3) a_{2\beta_2} \dot{b}_{1\beta_2} u_\alpha^2, \text{ where } \beta_1 \equiv b_1 b_2 b_3$$

$$= a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 + \tfrac{3}{4} (a_1 \dot{b}_1 b_2 b_3)(a_2 a_1 b_2 b_3) a_{2\beta_2} \dot{b}_{1\beta_2} u_\alpha^2$$

$$= a^2_{1\beta_1} a^2_{2\beta_2} u_\alpha^2 - \tfrac{3}{4} (AB)^2 b^2_{1\beta_2} u_\alpha^2$$

$$= a_\beta^2 a'^2_{\beta'} u_\alpha^2, \text{ since } (AB)^2 = 0.$$

3rd term $= a_{1\beta_1}u_{\beta_1}a^2_{2\beta_2}u_\alpha a_{1\alpha}$

$= \frac{1}{4}u^2_{\beta_1}a^2_{2\beta_2}a^2_{1\alpha}$

$= \frac{1}{4}a^2_\alpha a'^2_\beta u^2_{\beta'}.$

4th term $= a_{1\beta_1}a_{2\beta_1}a_{2\beta_2}u_{\beta_2}u_\alpha a_{1\alpha}$

$= \frac{1}{4}u_{\beta_1}a_{2\beta_1}a_{2\beta_2}u_{\beta_2}a^2_{1\alpha}$

$= \frac{1}{4}u^2_{\beta_1}a^2_{2\beta_2}a^2_{1\alpha} + \frac{1}{4}u_{\beta_1}\dot{a}_{2\beta_1}a_{2\beta_2}\dot{u}_{\beta_2}a^2_{1\alpha}$

$= \frac{1}{4}u^2_{\beta_1}a^2_{2\beta_2}a^2_{1\alpha} + \frac{1}{4}(ub_1b_2b_3)(a_2u\dot{b}_2\dot{b}_3)a_{2\beta_2}\dot{b}_{1\beta_2}a^2_{1\alpha}$

$= \frac{1}{4}u^2_{\beta_1}a^2_{2\beta_2}a^2_{1\alpha} + \frac{3}{4}(ub_1b_2b_3)(a_2ub_2b_3)a_{2\beta_2}b_{1\beta_2}a^2_{1\alpha}$

$= \frac{1}{4}u^2_{\beta_1}a^2_{2\beta_2}a^2_{1\alpha} + \frac{3}{16}(ua_2b_2b_3)(a_2ub_2b_3)\,b^2_{1\beta_2}a^2_{1\alpha}$

$= \frac{1}{4}a^2_\alpha a'^2_\beta u^2_{\beta'} - \frac{3}{16}(Ba'u)^2b^2_\beta a^2_\alpha.$

5th term $= a_{1\beta_1}u_{\beta_1}a_{2\beta_2}a_{1\beta_2}u_\alpha a_{2\alpha}$

$= \frac{1}{4}a_{1\beta_1}u_{\beta_1}u_{\beta_2}a_{1\beta_2}a^2_{2\alpha}$

$= \frac{1}{4}a^2_\alpha a'^2_\beta u^2_{\beta'} - \frac{3}{16}(Ba'u)^2b^2_\beta a^2_\alpha$ (as for the 4th term).

6th term $= a^2_{1\beta_1}a_{2\beta_2}u_{\beta_2}u_\alpha a_{2\alpha}$

$= \frac{1}{4}a_{1\beta_1}a^2_{2\alpha}u^2_{\beta_2}$

$= \frac{1}{4}a^2_\alpha a'^2_\beta u^2_{\beta'}.$

By combining the above six terms, we have $Z = -\frac{1}{16}a^2_\alpha b^2_\beta(Ba'u)^2$.

Thus, from (A.2)

$$\tfrac{4}{3}u^2_\alpha(a\,.\,A\beta\,.\,A'\beta'\,.\,u)^2 = \tfrac{16}{3}u^2_\alpha(X - 2Y + Z)$$

$$= -\tfrac{16}{3}u^2_\alpha[a^2_\beta a'^2_{\beta'}u^2_{\alpha'} - 2(\tfrac{1}{3}a^2_\beta a'^2_{\beta'}u^2_{\alpha'} - \tfrac{1}{12}a^2_{\alpha'}a'^2_\beta u^2_{\beta'}) - \tfrac{1}{16}a^2_\alpha b^2_\beta(Ba'u)^2]$$

(A.3) $$= \tfrac{16}{9}a^2_\beta a'^2_{\beta'}u^2_\alpha u^2_{\alpha'} + \tfrac{8}{9}a^2_\alpha a'^2_\beta u^2_{\alpha'}u^2_{\beta'} - \tfrac{1}{3}a^2_\alpha b^2_\beta u^2_{\alpha'}(Ba'u)^2.$$

From (A.1) and (A.3) condition (7.1) therefore reduces to

$$\tfrac{16}{9}a^2_\beta a'^2_{\beta'}u^2_\alpha u^2_{\alpha'} + \tfrac{16}{3}a^2_\alpha a'^2_{\alpha'}u^2_\beta u^2_{\beta'} + \tfrac{4}{9}a^2_\alpha a'^2_\beta u^2_{\alpha'}u^2_{\beta'} = \tfrac{16}{9}a^2_\beta a'^2_{\beta'}u^2_\alpha u^2_{\alpha'} + \tfrac{8}{9}a^2_\alpha a'^2_\beta u^2_{\alpha'}u^2_{\beta'} - \tfrac{1}{3}a^2_\alpha b^2_\beta u^2_{\alpha'}(Ba'u)^2,$$

i e. $$a^2_\alpha a'^2_\alpha u^2_\beta u^2_{\beta'} - 16a^2_\alpha a'^2_\beta u^2_{\alpha'}u^2_{\beta'} + 12a^2_\alpha b^2_\beta u^2_{\alpha'}(Ba'u)^2 = 0,$$

which gives $$\Delta^2\Sigma'^2 - 16\Delta\Theta'\Sigma\Sigma' + 12\Delta\Delta'\Sigma\chi' = 0.$$

The equations (7.2) determining the developable surface previously considered are therefore

$$\chi = 0 \quad \text{and} \quad \Delta\Sigma'^2 - 16\Theta'\Sigma\Sigma' + 12\Delta'\Sigma\chi' = 0.$$

To obtain the equations of the curve locus of the vertices of the ∞^1 Φ-tetrahedra it is merely necessary to reciprocate this developable with respect to f', since the vertices are the poles of the faces with respect to f'.

The equation of the polar reciprocal of the quartic envelope specified in (7.2) is thus given by substituting $b_x b$ for u in the latter equation.

The polar reciprocal of Σ is given by the following expression equated to zero:

$$\begin{aligned} b_x b_a b'_a b'_x &= b_a^2 b'^2_x + \dot{b}_x b_a \dot{b}'_a b'_x \\ &= b_a^2 b'^2_x - \tfrac{1}{2}(Bax)^2 \\ &= \Theta f' - \tfrac{1}{2} k'. \end{aligned}$$

For Σ', it is $$\begin{aligned} b_x b_\beta b'_\beta b'_x &= \tfrac{1}{4} b_\beta^2 b'^2_x \\ &= \tfrac{1}{4}\Delta' f', \end{aligned}$$

and for χ', $$\begin{aligned} b_x (Bab)\,(Bab')\,b'_x &= \tfrac{1}{3}\dot{b}_x (Bab)\,(\dot{B}ab')\,b'_x \\ &= \tfrac{1}{3} a_\beta^2 b'^2_x - \tfrac{1}{3} a_\beta b'_\beta a_x b'_x \\ &= \tfrac{1}{3} a_\beta^2 b'^2_x - \tfrac{1}{12} b'^2_\beta a_x^2 \\ &= \tfrac{1}{3}\Theta' f' - \tfrac{1}{12}\Delta' f. \end{aligned}$$

By substitution of these results in the second equation of (7.2) the quartic surface of the polar reciprocal with respect to f' is found to be $\frac{1}{16}\Delta^2\Delta'^2 f'^2 - 4\Delta\Delta'\Theta' f'\,(\Theta f' - \frac{1}{2}k') + 12\Delta\Delta'\,(\Theta f' - \frac{1}{2}k')\,(\frac{1}{3}\Theta' f' - \frac{1}{12}\Delta' f) = 0$, which reduces to $\Delta f'^2 - 16\Theta f f' + 8fk' = 0$, and so the equations determining the locus of vertices are, as in (7.3),

$$k = 0 \text{ and } \Delta f'^2 - 16\Theta f f' + 8fk' = 0.$$

For the second system of Φ-tetrahedra, (7.4) and (7.5) are true in similar manner.

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On Symmetric, Orthogonal, and Skew-Symmetric Matrices

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1. Introduction and Notation. In this paper all the scalars are real and all matrices are, if not stated to be otherwise, p-rowed square matrices. The diagonal and superdiagonal elements of a symmetric matrix, and the superdiagonal elements of a skew-symmetric matrix, will be called the distinct elements of the respective matrices. Σ will denote both the set of all symmetric matrices and the $\frac{1}{2}p\,(p+1)$-dimensional space whose coordinates are the distinct elements arranged in some specific order. K will denote both the set of all skew-symmetric matrices and the $\frac{1}{2}p\,(p-1)$-dimensional space whose coordinates are the distinct elements arranged in some specific order. Any sub-set of Σ (K) will mean both the sub-set of symmetric (skew-symmetric) matrices and the set of points of Σ (K). Any point function defined in Σ (K) will be written as a function of a symmetric (skew-symmetric) matrix. D_a will denote the diagonal matrix whose diagonal elements are $a_1, a_2, \ldots, a_p$. The characteristic roots of a symmetric matrix will be called its roots.

The orthogonal matrices Γ with $|\Gamma + I| \neq 0$ and the skew-symmetric matrices X are in (1, 1)-correspondence, on account of the following pair of equivalent equations:

$$\Gamma = 2(I+X)^{-1} - I, \tag{1}$$

$$X = 2(I+\Gamma)^{-1} - I. \tag{1'}$$

That the skew-symmetry of X implies the orthogonality of Γ, and vice versa,[1] is the direct result of the following computation:

$$\begin{aligned}[2(I+X)^{-1} - I]\ [2(I+X)^{-1} - I]' &= (I+X)^{-1}\ (I-X)\ \ (I+X)(I-X)^{-1} \\ &= (I+X)^{-1}(I+X)(I-X)(I-X)^{-1} = I,\end{aligned}$$

$$\begin{aligned}&[2(I+\Gamma)^{-1} - I] + [2(I+\Gamma)^{-1} - I]' \\ &= 2(I+\Gamma)^{-1} + 2(I+\Gamma)^{-1}\Gamma - 2I = 2(I+\Gamma)^{-1}(I+\Gamma) - 2I = 0.\end{aligned}$$

If $S \in \Sigma$, and if its roots are $\theta_1, \ldots, \theta_p$, it is well known that

$$S = \Delta D_\theta \Delta' \tag{2}$$

where Δ is some orthogonal matrix.

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[1] This result, together with a lengthy derivation, is given in Kowalewski, *Einführung in die Determinantentheorie* (Leipzig, 1909), pp. 171-175.

LEMMA 1. *If A is any $p \times p$ matrix, there is a matrix D_ϵ, with $\epsilon_i = +1$ or -1 $(i = 1, \ldots, p)$, such that $| A + D_\epsilon | \neq 0$.*

Proof. The lemma is evidently true for $p = 1$. Assume it is true for $p - 1$. Write

$$A = \begin{pmatrix} A_1 & b' \\ c & d \end{pmatrix},$$

where A_1 is $(p-1) \times (p-1)$ and b, c are rows. By assumption there is a D_η, with $\eta_i = +1$ or -1 $(i = 1, \ldots, p-1)$, such that $| A_1 + D_\eta | \neq 0$. Since

$$\begin{vmatrix} A_1 + D_\eta & b' \\ c & d+1 \end{vmatrix} - \begin{vmatrix} A_1 + D_\eta & b' \\ c & d-1 \end{vmatrix} = 2 \mid A_1 + D_\eta \mid \neq 0, \tag{3}$$

the two determinants on the left side of (3) cannot both vanish. Hence the lemma is proved.

LEMMA 2. *If $S \in \Sigma$, there is an orthogonal Γ with $| \Gamma + I | \neq 0$ such that $S = \Gamma D_\theta \Gamma'$.*

Proof. For the Δ in (2) there is, by Lemma 1, a D_ϵ with $| \Delta + D_\epsilon | \neq 0$. Hence $| \Delta D_\epsilon + I | \neq 0$. Moreover, ΔD_ϵ also satisfies (2). Hence the matrix $\Gamma = \Delta D_\epsilon$ answers all the requirements.

Combining Lemma 2 with (1) we obtain

THEOREM 1. *If $S \in \Sigma$, there is an $X \in K$ such that*

$$S = [2(I + X)^{-1} - I] \, D_\theta \, [2(I + X)^{-1} - I]'. \tag{4}$$

In § 2 we investigate the uniqueness of the expression (4) for a given S. (4) expresses each distinct element of S as a function of $\frac{1}{2}p(p+1)$ arguments, viz., the distinct elements of X and the θ's. In § 3 we evaluate the functional determinant of these functions or, to use another term, the Jacobian of the transformation (4). In § 4 we apply the results to prove a formula for the integral of some function of S.

2. The symmetric matrices S which have multiple roots satisfy the equation $f(S) = 0$, where $f(S)$ is the discriminant of the characteristic equation of S. Let F be the surface $f(S) = 0$. If $S \in \Sigma - F$,

we shall so name the roots $\theta_1, \ldots, \theta_p$ that $\theta_1 > \theta_2 > \ldots > \theta_p$. Thus each θ_i is a well-defined function of S. The equation

$$S = \Gamma D_\theta \Gamma', \tag{5}$$

where $\Gamma = (\gamma_{ij})$ is orthogonal, then uniquely determines every column of Γ except for a sign. In particular,

$$| \gamma_{1i} | = g_i(S) \qquad (i = 2, \ldots, p)$$

are well-defined functions of S. Let F_i be the surface $g_i(S) = 0$. If $S \in \Sigma - F - F_2 - \ldots - F_p$, we take $\gamma_{1i} = -g_i(S)$ $(i = 2, \ldots, p)$. Thus all the columns except the first one of Γ are uniquely determined, since the signs of the top elements of these columns are determined. Hence we have

LEMMA 3. *If $S \in \Sigma - F - F_2 - \ldots - F_p$, equation (5) has exactly two solutions for Γ: Γ_1 and Γ_2, whose elements (1, 2), (1, 3), ..., (1, p) are all negative. Γ_1 and Γ_2 differ only in the respect that the first column of one is the negative of the first column of the other.*

By virtue of Lemma 3 all the elements γ_{ij} $(i, j = 2, \ldots, p)$ of Γ are well-defined functions of S. Writing $\Gamma^{(1)} = (\gamma_{ij})_{(i,\, j = 2 \ldots p)}$ we have $| \Gamma^{(1)} + I | = h(S)$, a well-defined function of S. Let F' be the surface $h(S) = 0$. Let $E = \Sigma - F - F_2 - \ldots - F_p - F'$.

LEMMA 4. *If $S \in E$, one of the determinants $| \Gamma_1 + I |$ and $| \Gamma_2 + I |$ in Lemma 3 is zero while the other is not zero.*

Proof. By Lemma 3 and the defiinition of E we have

$$| \Gamma_1 + I | + | \Gamma_2 + I | = 2 | \Gamma^{(1)} + I | \neq 0.$$

Hence one of the determinants is not zero. Suppose $| \Gamma_1 + I | \neq 0$. Then, by (1), $\Gamma_1 = 2(I + X)^{-1} - I$, where X is skew-symmetric. Letting H be the diagonal matrix $[-1, 1, \ldots, 1]$ we have $\Gamma_2 = \Gamma_1 H$. Hence

$$\begin{aligned} | \Gamma_2 + I | = | \Gamma_1 H + I | &= | 2(I + X)^{-1} H + I - H | \\ &= | X + I |^{-1} | 2H + (I + X)(I - H) | . \end{aligned}$$

It is easily seen that the first row of the matrix $2H + (I + X)(I - H)$ consists of zeros. Hence $| \Gamma_2 + I | = 0$. This completes the proof.

If we take Γ to be Γ_1 or Γ_2 according as $| \Gamma_1 + I | \neq 0$ or $| \Gamma_2 + I | \neq 0$ we obtain

LEMMA 5. *If $S \in E$, there is a unique Γ which satisfies (5) and the following conditions:*

(i) $\gamma_{1i} < 0 \qquad (i = 2, \ldots, p)$

(ii) $| \Gamma + I | \neq 0.$

By combining Lemma 5 with (1) and noticing that in (1) the non-diagonal elements of Γ and $(I + X)^{-1}$ must have the same sign, we obtain

THEOREM 2. *If $S \in E$, the equation* (4) *has a unique solution in X such that the elements* $(1, 2), (1, 3), \ldots, (1, p)$ *of $(I + X)^{-1}$ are all negative.*

Let M be the sub-set of K defined by the condition that the elements $(1, 2), (1, 3), \ldots, (1, p)$ of $(I + X)^{-1}$ are all negative, let Θ be the sub-set of the space of $(\theta_1, \ldots, \theta_p)$ defined by the condition $\theta_1 > \theta_2 > \ldots > \theta_p$, and let $E^* = M \times \Theta$, a sub-set of the $\frac{1}{2}p(p+1)$-dimensional space. Theorem 2 asserts that the equation (4) effects a $(1, 1)$-mapping of E^* on E. Notice also that by the definition of E, it differs from Σ by a set of measure zero.

3. In order to facilitate the computation of the functional determinant[1] for (4) we shall adopt the following notation. If $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ and if

$$x_i = \sum_{j=1}^{m} a_{ij} y_j \qquad (i = 1, \ldots, m),$$

then we define the symbol $D(x; y)$ to be the discriminant $| a_{ij} |$.

LEMMA 6.

(i) $$D(x; y) = \frac{1}{D(y; x)}$$

(ii) $$D(x; y) = D(x; z)\, D(z; y).$$

Proof. (i) is an immediate consequence of definition; (ii) is a special case of the multiplicative law of functional determinants.

The extension of (ii) to more than two factors is obvious.

In the equation

(6) $$X = AYA',$$

X is symmetric or skew-symmetric according as Y is symmetric or skew-symmetric. In either case (6) expresses each distinct element of X as a linear function of the distinct elements of Y, with coefficients depending on A. The following lemma gives the discriminant $D(X; Y)$ of this set of linear functions.

---

[1] Functional determinants are considered as determined up to a sign. In all the computations in this section signs of functional determinants are neglected.

LEMMA 7. *For* (6) *we have*

$$D(X;\ Y) = \begin{cases} |A|^{p+1} & \text{if } Y \text{ is symmetric} \\ |A|^{p-1} & \text{if } Y \text{ is skew-symmetric.} \end{cases} \tag{7}$$

Proof. It is sufficient to prove (7) for a non-singular A. Now every non-singular matrix is a product of a finite number of matrices of either of the following two types, (a) a diagonal matrix whose diagonal elements are 1 with the exception of one, which is, say, a; (b) a matrix whose diagonal elements are 1 and whose non-diagonal elements are 0 with the exception of one. Further, if we denote $D(X;\ Y)$ for (6) by $P(A)$, a moment of reflection will show that $P(AB) = P(A)\,P(B)$. Hence it is sufficient to prove (7) for A belonging to either of the types (a) and (b), i.e. $D(X;\ Y) = \begin{cases} a^{p+1} \\ a^{p-1} \end{cases}$ if A is of type (a) and $D(X;\ Y) = 1$ for both cases if A is of type (b). The proof of these assertions is easy and is left to the reader.

If $x_1, \ldots, x_m$ are functions of $y_1, \ldots, y_m$, then by definition the functional determinant is equal to $D(dx;\ dy)$, where dx denotes the system $dx_1, \ldots, dx_m$. For brevity we write dA for the matrix whose elements are the differentials of the elements of A.

In order to differentiate (1) we use the formula

$$dA^{-1} = -A^{-1}(dA)A^{-1}.$$

Then

$$d\Gamma = -2(I+X)^{-1}\,dX\,(I+X)^{-1} = -\tfrac{1}{2}(I+\Gamma)\,dX\,(I+\Gamma). \tag{8}$$

Let J be the functional determinant for (4), i.e. for

$$S = \Gamma D_\theta \Gamma' \tag{9}$$

where Γ is given by (1). Differentiating (9) and using (8) we have

$$\begin{aligned} dS = \Gamma D_\theta d\Gamma' + (d\Gamma)\,D_\theta\Gamma' + \Gamma D_\theta \Gamma' = \tfrac{1}{2}\Gamma D_\theta (I+\Gamma')\,dX\,(I+\Gamma') \\ - \tfrac{1}{2}(I+\Gamma)\,dX\,(I+\Gamma)\,D_\theta\Gamma' + \Gamma D_{d\theta}\Gamma'. \end{aligned} \tag{10}$$

Also,

$$J = D(dS;\ dX,\ d\theta). \tag{11}$$

From (10) we get

$$\Gamma' dS\,\Gamma = \tfrac{1}{2}D_\theta(I+\Gamma')\,dX\,(I+\Gamma) - \tfrac{1}{2}(I+\Gamma')\,dX\,(I+\Gamma)\,D_\theta + D_{d\theta}. \tag{12}$$

If we write ξ_i for $d\theta_i$, and define U and Y by

$$U = \Gamma' dS\,\Gamma \tag{13}$$

$$Y = (I+\Gamma')\,dX\,(I+\Gamma), \tag{14}$$

then (12) gives

$$U = \tfrac{1}{2} D_\theta Y - \tfrac{1}{2} Y D_\theta + D_\xi,$$

i.e.,

$$\begin{aligned} u_{ii} &= \xi_i \qquad (i = 1, \ldots, p), \\ u_{ij} &= \tfrac{1}{2}(\theta_i - \theta_j)\, y_{ij} \qquad (i < j). \end{aligned} \tag{15}$$

(13), (14) and (15) give respectively

$$D(dS;\, U) = \frac{1}{D(U;\, dS)} = 1, \text{ by Lemmas 6 and 7;}$$

$$D(Y;\, dX) = 2^{p(p-1)} \mid I + X \mid^{-(p-1)}, \text{ by Lemma 6 and (1);}$$

$$D(U;\, Y, \xi) = 2^{-\frac{1}{2}p(p-1)} \prod_{i<j} (\theta_i - \theta_j).$$

Hence, by (11) and Lemma 7,

$$\begin{aligned} J &= D(dS;\, U)\, D(U;\, Y, \xi)\, D(Y, \xi;\, dX, \xi) \\ &= D(dS;\, U)\, D(U;\, Y, \xi)\, D(Y;\, dX), \end{aligned}$$

and so, finally,

$$J = 2^{\frac{1}{2}p(p-1)} \mid I + X \mid^{-(p-1)} \prod_{i<j} (\theta_i - \theta_j). \tag{16}$$

4. Let us first evaluate the following integrals:

$$A_p = \int_K \mid I + X \mid^{-(p-1)} dm \tag{17}$$

$$B_p = \int_M \mid I + X \mid^{-(p-1)} dm \tag{18}$$

where X is a p-rowed skew-symmetric matrix, dm denotes the volume element, K is the $\frac{1}{2}p(p-1)$-dimensional space, and M is defined in the last paragraph of § 2.

We write

$$X = \begin{pmatrix} 0 & x \\ -x' & Y \end{pmatrix}, \qquad x = (x_1, \ldots, x_{p-1})$$

so that

$$\mid I + X \mid = \begin{vmatrix} 1 & x \\ -x' & I + Y \end{vmatrix} = \mid I + Y \mid (1 + x(I + Y)^{-1} x').$$

Also, the elements $(1, 2), (1, 3), \ldots, (1, p)$ of $(I+X)^{-1}$ are the elements of the row $-(1 + x(I + Y)^{-1} x')^{-1} x (I + Y)^{-1}$. Hence M is the set of points such that every element of the row $x(I + Y)^{-1}$ is positive. Making the transformation

$$x' = (I + Y)\, u', \qquad u = (u_1, \ldots, u_{p-1})$$

whose Jacobian is $\mid I + Y \mid$, we obtain

$x(I+Y)^{-1}x' = u(I+Y)u' = uu' = u_1^2 + \ldots + u_{p-1}^2$.

Hence

$$(19)\quad A_p = \int_{K_1} |I+Y|^{-(p-2)}\,dm \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (1+u_1^2+\cdots + u_{p-1}^2)^{-(p-1)}\,du_1\ldots du_{p-1},$$

$$(20)\quad B_p = \int_{K_1} |I+Y|^{-(p-2)}\,dm \int_{0}^{\infty}\cdots\int_{0}^{\infty} (1+u_1^2+\cdots + u_{p-1}^2)^{-(p-1)}\,du_1\ldots du_{p-1},$$

where K_1 is the $\frac{1}{2}(p-1)(p-2)$-dimensional space. It follows from (19) and (20) that

$$(21)\quad B_p = 2^{-(p-1)}A_p,$$

$$(22)\quad A_p = A_{p-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(1+u_1^2+\ldots+u_{p-1}^2)^{-(p-1)}du_1\ldots du_{p-1}$$
$$= \frac{\pi^{p/2}}{2^{p-2}\,\Gamma(p/2)}A_{p-1},$$

and the easy computation $A_2 = \pi$ leads to the result

$$(23)\quad A_p = \pi\prod_{r=3}^{p}\frac{\pi^{r/2}}{2^{r-2}\,\Gamma(r/2)} = \frac{\pi^{p(p+1)/4}}{2^{(p-1)(p-2)/2}\prod_{r=1}^{p}\Gamma(r/2)},$$

whence also

$$(24)\quad B_p = \frac{\pi^{p(p+1)/4}}{2^{p(p-1)/2}\prod_{r=1}^{p}\Gamma(r/2)}.$$

We can now prove

THEOREM 3. *Let $f(S)$ be a function of the distinct elements of the p-rowed symmetric matrix S, such that $f(S)$ is a function only of the roots $\theta_1, \ldots, \theta_p$ of S, $\theta_1 \geqq \theta_2 \geqq \ldots \geqq \theta_p$:*

$$f(S) = g(\theta_1, \ldots, \theta_p).$$

Let Σ be the $\frac{1}{2}p(p+1)$-dimensional space. Then

$$(25)\quad \int_{\Sigma} f(S)\,dm = \frac{\pi^{p(p+1)/4}}{\prod_{r=1}^{p}\Gamma(r/2)}\int_{\Theta} g(\theta_1,\ldots,\theta_p)\prod_{i<j}(\theta_i-\theta_j)\,d\theta_1\ldots d\theta_p,$$

where Θ is the domain $\theta_1 > \theta_2 > \ldots > \theta_p$.

Proof. By the remark at the end of §2 we have

$$\int_{\Sigma} f(S)\,dm = \int_{E} f(S)\,dm.$$

By Theorem 2 we may use the transformation (4), with the Jacobian (16). Hence

$$\int_{\Sigma} f(S)\,dm = 2^{\frac{1}{2}p(p-1)} \int_{E^*} |\,I + X\,|^{-(p-1)}\, g(\theta_1, \ldots, \theta_p) \prod_{i<j} (\theta_i - \theta_j)\,dm$$

where $E^* = M \times \Theta$. Hence

$$\int_{\Sigma} f(S)\,dm = 2^{\frac{1}{2}p(p-1)} B_p \int_{\Theta} g(\theta_1, \ldots, \theta_p) \prod_{i<} (\theta_i - \theta_j)\,d\theta_1, \ldots d\theta_p.$$

Using (24) we get (25).

THEOREM 3 implies the following theorem on the probability distribution of roots, which is an important subject in statistics.

THEOREM 4. *Let the distinct elements of the p-rowed symmetric matrix S be random variables whose joint distribution in Σ has a probability density $f(S)$, such that $f(S)$ is a function only of the roots $\theta_1 \geqq \ldots \geqq \theta_p$ of S:*

$$f(S) = g(\theta_1, \ldots, \theta_p).$$

Then the joint distribution of roots in Θ has the probability density function

$$\frac{\pi^{p(p+1)/4}}{\prod_{r=1}^{p} \Gamma(r/2)}\, g(\theta_1, \ldots, \theta_p) \prod_{i<j} (\theta_i - \theta_j).$$

Proof. If A is any sub-set of Θ, let B be the sub-set of Σ such that $(\theta_1, \ldots, \theta_p) \in A$. Then

$$\Pr\{(\theta_1, \ldots, \theta_p) \in A\} = \int_{\Sigma} \phi(S) f(S)\,dm, \tag{26}$$

where $\phi(S) = 1$ or 0 according as $S \in B$ or $S \bar{\in} B$. Now $\phi(S) = \psi(\theta_1, \ldots, \theta_p)$, where $\psi = 1$ or 0 according as $(\theta_1, \ldots, \theta_p) \in A$ or not. Hence $\phi(S) f(S) = \psi(\theta_1, \ldots, \theta_p) g(\theta_1, \ldots, \theta_p)$. Applying Theorem 3 to (26) we get

$$\Pr\{(\theta_1, \ldots, \theta_p) \in A\}$$

$$= \frac{\pi^{p(p+1)/4}}{\prod_{r=1}^{p} \Gamma(r/2)} \int_{\Theta} \psi(\theta_1, \ldots, \theta_p)\, g(\theta_1, \ldots, \theta_p) \prod_{i<j} (\theta_i - \theta_j)\,d\theta_1 \ldots d\theta_p$$

$$= \frac{\pi^{p(p+1)/4}}{\prod_{r=1}^{p} \Gamma(r/2)} \int_{A} g(\theta_1, \ldots, \theta_p) \prod_{i<j} (\theta_i - \theta_j)\,d\theta_1 \ldots d\theta_p,$$

which proves the theorem.

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Some Remarks on Uniform Convergence

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Introduction.

A useful test for uniform convergence is that first established by Buchanan and Hildebrandt [4] which is as follows.

(A) "If a sequence $f_n(x)$ of monotonic functions converges to a continuous function $f(x)$ in $[a, b]$ then this convergence is uniform."

In §1 of this paper it is shown that this test is included in a sequence of theorems, each of which establishes a type of uniform convergence. The first is a well-known topological theorem on limit sets, the second is a result on the limits of rectifiable arcs, the third is a generalisation of (A) due to Behrend [3], the fourth is (A) itself, the fifth is a one-sided version of Bendixson's test and the sixth is Bendixson's test.

In §2 these theorems are partially extended to theorems of more than one variable.

A similar result to (A) in which the range $[a, b]$ is replaced by $(-\infty, \infty)$ has been established by Pólya [8] and extended by Conti [5]. There are corresponding extensions of the theorems proved here. The proof of Theorem 2 is related to the ideas used in papers by Radó and Reichelderfer [9], Ayer [1], Tsuji [11], and Ayer and Radó [2]. The paper by Goodstein [6] is also concerned with a theorem similar to (A).

(A) may be stated in the alternative form

"If a sequence of additive non-negative functions $F_n(x)$ defined for all measurable subsets of $[a, b]$ converges to an absolulely continuous limit function $F(x)$, then the convergence is uniform."

An example of the use of the alternative form of (A) is given in Scheffé [10].

§1.

Let S be a compact metric space and ρ denote the distance function in S. For any subset X of S let $\mathcal{U}(X, \epsilon)$ denote the set of all points whose distance from X is less than ϵ, ϵ being a positive number.

Theorem 1. *Let X_n be any sequence of sets in S; then for every $\epsilon > 0$ there exists an integer N such that for $n \geq N$*

$$\mathcal{U}(\overline{\lim}\, X_n, \epsilon) \supset X_n$$

where $\overline{\lim}\, X_n$ is the upper topological limit of X_n.

For suppose that the contrary were the case. Then for some positive number ϵ_0 there is a sequence of integers n_i such that each X_{n_i} contains a point x_{n_i} and

$$\lim_{i\to\infty} n_i = \infty, \qquad \rho(x_{n_i}, \overline{\lim}\, X_{n_i}) \geqq \epsilon_0. \tag{1}$$

As S is compact, a subsequence of the x_{n_i} tends to a point p. By (1)

$$\rho(p, \overline{\lim}\, X_n) \geqq \epsilon_0. \tag{2}$$

But by the definition of $\overline{\lim}\ X_n$, $p \subset \overline{\lim}\ X_n$, a contradiction of (2)

THEOREM 2. *Let S be a bounded portion of Euclidean space of any (finite) number of dimensions and X_n be a sequence of arcs in S. Suppose that the length of X_n is l_n. If $\lim_{n\to\infty} l_n = l$ and there is an arc X of length l contained in $\overline{\lim}\ X_n$, then for every $\epsilon > 0$ there exists an integer N such that for $n \geqq N$ and x_n belonging to X_n,*

$$\rho(x_n, X) < \epsilon.$$

By Theorem 1 it is sufficient to show that

$$X = \overline{\lim}\ X_n.$$

If this is not the case then there is a point y contained in $\overline{\lim}\ X_n - X$. As X is a closed set $\rho(y, X) = \eta > 0$. There is a positive number δ such that $\eta > 8\delta$ and such that every polygonal line joining the two end-points of X, whose vertices belong to X and whose segments are of length less than or equal to δ, has total length greater than

$$l - \eta/4.$$

Such a polygonal line is constructed as follows. One end-point of the arc X, say p_0, is selected and called the first point, the other end-point is called the last point p', and the other points are ordered in accordance with this nomenclature.

p_0 is the first point of the polygonal line. p_1 is p' if $\rho(p_0, p') \leqq \delta$, otherwise it is the last point after p_0 whose distance from p_0 is δ.

Generally, p_{i+1} is p' if $\rho(p_i, p') \leqq \delta$, otherwise it is the last point after p_i for which $\rho(p_i, p_{i+1}) = \delta$. Let $p_0 p_1 \ldots p_{t-1}\, p'$ be the polygonal line so formed and write $p' = p_t$.

The circles $C(p_i, \delta/2)$, $i = 0, 1, \ldots, t$, of centre p_i and radius $\delta/2$, are non-overlapping and

$$(t+1)\,\delta > l - \eta/4. \tag{3}$$

As $p_0, p_1, \ldots, p_t$ belong to $\underline{\lim}\ X_n$, there is an integer N_0 such that

$$\rho(p_i, X_n) \leqq \eta/8(t+1), \qquad i = 0, \ldots, t, \qquad n \geqq N_0.$$

If the circle $C(p_i, \delta/2)$ does not contain an end-point of X_n, the part of X_n contained in it has length at least $\delta - \eta/4(t+1)$. If it does contain an end-point the length contained is at least $\delta/2 - \eta/8(t+1)$.

X_n has two end-points; thus the set of circles $C(p_i, \delta/2)$, $i=0, \ldots, t$ contain a part of X_n of length at least

$$t(\delta - \eta/4(t+1)) > l - \eta/2 - \delta > l - 5\eta/8. \tag{4}$$

Now $y \subset \overline{\lim}\ X_n$, and thus for some integer n arbitrarily large, say $n = m \geqq N_0$,

$$\rho(y, X_m) < \eta/4.$$

The circle $C(y, \eta - \delta/2)$ does not overlap any of the $C(p_i, \delta/2)$ and contains a part of X_n of length greater than or equal to

$$\eta - \delta/2 - \eta/4 > 11\eta/16. \tag{5}$$

By (4), (5)

$$l_m \geqq l + \eta/16.$$

Since m is arbitrarily large this is in contradiction with the fact that $\lim_{n\to\infty} l_n = l$.

COROLLARY. *Let $f_n(x)$ be a set of continuous functions defined over $a \leqq x \leqq b$ such that*

(i) $\lim_{n\to\infty} f_n(x) = f(x), \qquad a \leqq x \leqq b,$

(ii) $$\lim_{n\to\infty} \left\{ \lim_{h\to 0} \int_a^b \left[1 + \left(\frac{f_n(x+h) - f_n(x)}{h}\right)^2\right]^{\frac{1}{2}} dx \right\}$$
$$= \lim_{h\to 0} \int_a^b \left[1 + \left(\frac{f(x+h) - f(x)}{h}\right)^2\right]^{\frac{1}{2}} dx,$$

(iii) *$f(x)$ is continuous.*

Then the convergence of the sequence $f_n(x)$ to $f(x)$ is uniform in x, $a \leqq x \leqq b$.

Denote by X_n the set of points in the (x, y) Euclidean plane with coordinates $(x, f_n(x))$, $a \leqq x \leqq b$, and by X the set of points $(x, f(x))$, $a \leqq x \leqq b$. X_n and X are arcs with lengths, say l_n and l.

Condition (i) implies that $X \subset \underline{\lim}\ X_n$ and condition (ii) that $\lim_{n\to\infty} l_n = l$. Thus by Theorem 2 for a given $\epsilon > 0$,

$$X_n \subset \mathcal{U}(X, \epsilon) \qquad n \geqq N = N(\epsilon). \tag{6}$$

For each point p of X form the segment whose midpoint is p and which is parallel to the y-axis and is of length δ. Let the set of points formed from all such segments be called $Y(X, \delta)$. Because $f(x)$ is continuous, for a given $\delta > 0$ there is an $\epsilon > 0$ such that

$$\mathcal{U}_1(X, \epsilon) \subset Y(X, \delta) \tag{7}$$

where $\mathcal{U}_1(X, \epsilon)$ denotes that part of $\mathcal{U}(X, \epsilon)$ lying in the closed strip $a \leqq x \leqq b$.

The conclusion now follows from (6) and (7).

Remark. The continuity condition on the $f_n(x)$, $f(x)$ can be considerably relaxed: see e.g. Saks, *Theory of the integral*, p. 184.

THEOREM 3. *If $f_n(x)$, $f(x)$ are monotonic in $[a, b]$ then $f_n(x) \to f(x)$ uniformily in $[a, b]$ if and only if* (i) *$f_n(x) \to f(x)$ for all x of an everywhere dense set E in $[a, b]$ containing the set D of all discontinuities of $f(x)$, and*

(ii) $f_n(x - 0) \to f(x - 0)$, $\quad f_n(x + 0) \to f(x + 0)$ *for all x in D.*

The necessity of the conditions is trivial. To prove their sufficiency the given conditions are used to reduce this theorem to the case of continuous $f(x)$ which is discussed in Theorem 4.

Write the discontinuities of $f(x)$ as $c_1, c_2, \ldots$. Define the functions

$$\phi(x) = \sum_{c_k < x} (f(c_k) - f(c_k - 0)), \quad \phi_n(x) = \sum_{c_k < x} (f_n(c_k) - f_n(c_k - 0)),$$

$$\psi(x) = \sum_{c_k < x} (f(c_k + 0) - f(c_k)), \quad \psi_n(x) = \sum_{c_k < x} (f_n(c_k + 0) - f_n(c_k)),$$

$$g(x) = f(x) - \phi(x) - \psi(x), \qquad g_n(x) = f_n(x) - \phi_n(x) - \psi_n(x).$$

Then $g(x)$, $g_n(x)$ are monotone and $g(x)$ is continuous. Conditions (i) and (ii) imply that

$$\phi_n(x) \to \phi(x), \qquad \psi_n(x) \to \psi(x)$$

uniformly in $[a, b]$. Hence

$$g_n(x) \to g(x).$$

By the next theorem this convergence is uniform in x.

It follows that $f_n(x) \to f(x)$ uniformly in x, $a \leqq x \leqq b$.

THEOREM 4. (*Buchanan and Hildebrandt*) *If a sequence $f_n(x)$ of monotonic functions converges to a continuous function in $a \leqq x \leqq b$, say $f(x)$, then the convergence is uniform.*

An auxiliary sequence of functions $F_i(x)$ is defined as follows.

$$F_i(x_k) = f(x_k) \qquad \text{where } x_k = a + (b-a)\,k/i, \qquad k = 0, 1, 2, \ldots i,$$

and for other x of $a \leqq x \leqq b$, $F_i(x)$ is defined by linear interpolation.

Then $F_i(x) \to f(x)$, uniformly by Theorem 2. Also

$$\operatorname*{Sup}_{a<x<b} |f_n(x) - F_i(x)| \leqq \operatorname*{Max}_{k=0,\ldots,i-1} \{ |f_n(x_k) - F_i(x_{k+1})| + |f_n(x_{k+1}) - F_i(x_k)| \}.$$

Thus $f_n(x) \to f(x)$ uniformly.

THEOREM 5. *Let $f_n(x)$ be a sequence of continuous functions for each of which the upper right-hand derivative $f_n^+(x)$ satisfies* (i) $f_n^+(x) > -\infty$ *at every point x, $a \leqq x \leqq b$, except at most at those of an enumerable set,* (ii) $f_n^+(x) \geqq g(x)$ *for almost all x, where $g(x)$ is Perron-integrable over $a \leqq x \leqq b$.*

Suppose also that $f_n(x)$ tends to a continuous limit function $f(x)$. Then

$$f_n(x) \to f(x) \text{ uniformly in } x,\ a \leqq x \leqq b.$$

Write $\phi_n(x) = f_n(x) - \int_a^x g(x)\,dx$ where the integral sign denotes the Perron integral. Conditions (i) and (ii) imply that,

$$\phi_n(x+h) - \phi_n(x) = f_n(x+h) - f_n(x) - \int_x^{x+h} g(x)\,dx \geqq 0$$

for $a \leqq x < x + h \leqq b$.

By Theorem 4, $\phi_n(x) \to f(x) - \int_a^x g(x)\,dx$ uniformly in x. Thus $f_n(x) \to f(x)$ uniformly in x, $a \leqq x \leqq b$.

THEOREM 6. (*Bendixson*). *If $f_n(x)$ is a sequence of differentiable functions in $a \leqq x \leqq b$ and $|f_n'(x)| < k$ for all n and x, then if $f_n(x) \to f(x)$ as $n \to \infty$ it does so uniformly in x, $a \leqq x \leqq b$.*

By Theorem 5 all that needs to be proved is that the limit function $f(x)$ is continuous. If this were not so there would be a point x_0 and a sequence of points $x_1, x_2, \ldots$ such that

$$\lim_{i\to\infty} x_i = x_0, \quad \lim_{i\to\infty} f(x_i) \neq f(x_0).$$

Thus there are two points x_0, x_k such that

$$|(f(x_0) - f(x_k))/(x_0 - x_k)| > K.$$

For n sufficiently large

$$|(f_n(x_0) - f_n(x_k))/(x_0 - x_k)| > K.$$

This is not possible because of the condition $|f_n'(x)| < K$.

Thus the theorem is established.

§2.

Of the preceding theorems, 2 and 3 do not extend directly to the case of functions of more than one variable. This is related to the fact that the behaviour of the areas of a sequence of surfaces is more complicated than that of the lengths of a sequence of arcs. The analogues of Theorems 4 and 6 are true but now require direct proofs. Similar direct proofs hold for Theorems 4 and 6.

THEOREM 7. *Let $f_n(x_1, x_2, \ldots, x_r)$ be a sequence of functions of r variables $x_1, x_2, \ldots, x_r$, defined over the ranges $a_i \leqq x_i \leqq b_i$, $i = 1, 2, \ldots, r$, such that each function is monotone in each of the $x_2, x_3, \ldots, x_r$ variables separately when all the other variables are fixed. Suppose also that the sequence converges to a continuous limit function $f(x_1, x_2, \ldots, x_r)$, and that this convergence is uniform in x_1 when $x_2, x_3, \ldots, x_r$ are kept fixed. Then the convergence is uniform in all the variables $x_1, x_2, \ldots, x_r$ simultaneously.*

Since $f(x_1, x_2, \ldots, x_r)$ is continuous, for each $\epsilon > 0$ there is an integer $N = N(\epsilon)$ such that

$$|f(x_1', x_2', \ldots, x_r') - f(x_1, x_2, \ldots, x_r)| < \epsilon \tag{8}$$

provided $|x_i - x_i'| < 1/N$, $a_i \leqq x_i \leqq b_i$, $a_i \leqq x_i' \leqq b_i$, $i = 1, 2, \ldots, r$.

For each $\epsilon > 0$, there is an integer $n_0(\epsilon)$ such that

$$\begin{aligned} |f_n(x_1, a_2 + (b_2 - a_2)j_2/N, \ldots, a_r + (b_r - a_r)j_r/N) \\ - f(x_1, a_2 + (b_2 - a_2)j_2/N, \ldots, a_r + (b_r - a_r)j_r/N)| < \epsilon \end{aligned} \tag{9}$$

where $j_2, j_3, \ldots, j_r$ range over the integers $0, 1, \ldots, N$ independently and $a_1 \leqq x_1 \leqq b_1$, $n \geqq n_0(\epsilon)$.

Consider now any point $(x_1, x_2, \ldots, x_r)$ where $a_i \leqq x_i \leqq b_i$, $i = 1, 2, \ldots, r$. There are two sets of integers $j_2, j_3, \ldots, j_r$, and $j_2', j_3', \ldots, j_r'$, such that

$$\left.\begin{aligned} a_i + (b_i - a_i)j_i/N \leqq x_i < a_i + (b_i - a_i)j_i'/N \\ j_i' = j_i + 1,\ 0 \leqq j_i < N - 1 \end{aligned}\right\} i = 2, 3, \ldots, r.$$

Since the function is monotone

$$f_n(x_1, y_2, \ldots, y_r) \geqq f_n(x_1, x_2, \ldots, x_r) \geqq f_n(x_1', y_2', \ldots, y_r'),$$

where $y_i = a_i + (b_i - a_i)j_i/N$ or $y_i = a_i + (b_i - a_i)j_i'/N$ and similarly for y_i'. (The particular value depends on whether the function is increasing or decreasing in the i^{th} variable at the particular values of the other variables concerned and it may vary with n.)

By (8) and (9), for $n \geqq n_0(\epsilon)$,

$$| f_n(x_1, x_2, \ldots, x_r) - f(x_1, x_2, \ldots, x_r) | < 2\epsilon.$$

This proves the theorem.

Remarks.

(i) This theorem contains the direct analogue of Theorem 4 because if the functions are monotonic in x_1 then the convergence with respect to x_1 is necessarily uniform.

(ii) This theorem also contains a more general result in which the condition on $f_n(x_1, x_2, \ldots, x_r)$ as a function of x_1 is that of the corollary to Theorem 2.

(iii) A similar conclusion to that of the theorem is true if the conditions on the sequence are "monotone in $r - s$ variables and uniformly convergent simultaneously in the other s variables." This is not however true if the convergence is given to be uniform with respect to the s variables separately.

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